Unital Quantum Channels

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Definition (Unital Quantum Channel)

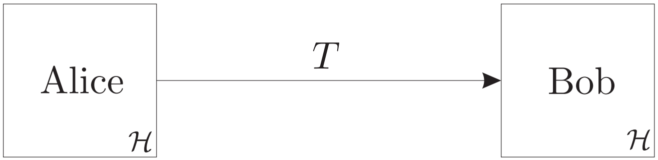

Figure: A quantum channel between Hilbert spaces of the same dimension

Definition

A quantum channel $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear, completely positive and trace preserving map. T is called unital if $T(\mathbb{1}) = \mathbb{1}$.

Jamiolkowski Isomorphism

Let T be a quantum channel with Kraus operator representation $T = \sum_k A_k \cdot A_k^\dagger$. Setting

$$\rho_T := (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) = \sum_k |e_k\rangle\langle e_k|\,,$$

$$|e_k\rangle := \frac{1}{\sqrt{d}} \sum_i (A_k |i\rangle)\,|i\rangle$$

gives a convex-linear isomorphism between the set of unital quantum channels and

$$\{\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \,:\, \rho \geq 0,\ \mathrm{tr}_1\rho = \mathrm{tr}_2\rho = \mathbb{1}/d\}\,.$$

Characterizing Unital Channels

Let $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a quantum channel, then the following are equivalent:

- T is unital, i.e. $T(\mathbb{1}) = \mathbb{1}$
- T is contractive with respect to the p-Schatten norm for every $p \in (1, \infty]$, that is, $\|T\|_{p-p} \leq 1$
- $\|T\|_{p-p} \leq 1$ for some $p \in (1, \infty]$ [PGWPR06]
- T can be represented as a convex combination of unitary maps on the bipartite system, i.e. $\hat{T} = \sum_i p_i \hat{U}_i$ with $U_i \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ unitary
- T is an affine-linear combination of unitary channels, $T(\rho) = \sum_i \lambda_i U_i \rho U_i^\dagger$ with $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$
- The asymptotic environment-assisted capacity of T obtains its maximum, $C_{\text{e.a.}}(T) = \max_\rho \min \{S(\rho), S(T(\rho))\} \stackrel{!}{=} \log d$ [SVW05][Win05]

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Extremal Unital Channels [LS93] [Cho75]

Kraus operator representation $T = \sum_k A_k \cdot A_k^\dagger$.

Theorem

T is extreme in the set of quantum channels if and only if $\left\{A_k^\dagger A_l\right\}_{k,l}$ is linearly independent. If T is unital, then T is extreme in the set of unital quantum channels if and only if

$$\left\{A_k^\dagger A_l \oplus A_l A_k^\dagger\right\}_{k,l}$$

is linearly independent.

$\exists$ extremal unital channels which are not extreme in the set of quantum channels? (Numerics for $d = 3$ and 4 Kraus operators $\rightsquigarrow$ yes!)

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Unitary Channels

Definition

A quantum channel T is called a unitary channel if

$$T(\rho) = U\rho U^\dagger \quad \text{with} \quad U \in \mathcal{B}(\mathcal{H}) \text{ unitary.}$$

In particular, every unitary channel is unital.

What is the convex hull of these channels? I.e. given a unital quantum channel T, can T be decomposed into

$$T(\rho) = \sum_i p_i \, U_i \rho U_i^\dagger \quad ?$$

Physical motivation: classical error mechanisms.

Classical Birkhoff's Theorem

Classical probability vector p, stochastic evolution matrix E

$$p' = E\,p.$$

E is called doubly stochastic (quantum analogue: unital) if and only if $E\,\mathbb{1} = \mathbb{1}$, i.e. all rows sum to 1.

Birkhoff's Theorem

The extremal doubly stochastic matrices are precisely the permutations. Hence every doubly stochastic matrix is a convex combination of permutations:

$$E = \sum_i p_i\,P_i, \quad P_i \text{ permutation matrix } \forall\, i.$$

The P_i are the invertible elements (quantum analogue: unitary channels).

Quantum Analogue of Birkhoff's Theorem [LS93]

Proposition

Let $d := \dim \mathcal{H} = 2$. Then every unital quantum channel T is a convex combination of unitary channels.
This holds no longer true for $d \geq 3$, i.e. for d odd and the Werner-Holevo channel

$$T_{WH} : \rho \mapsto \frac{1}{d-1} \left(\mathrm{tr}[\rho]\, \mathbb{1} - \rho^T \right).$$

But: asymptotic version for $T^{\otimes k}$ as $k \to \infty$ might be true! [GW02] [SVW05] [Win05]

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Environment Assisted Error Correction [GW02]

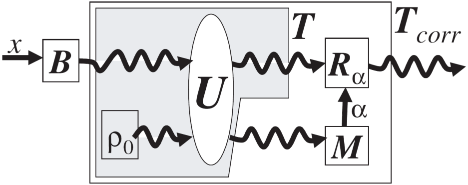

Figure: Correction scheme for a noisy channel T

Proposition

There exists a family of channels R_α restoring quantum information if and only if T is a convex combination of unitary channels.

Reformulating the Convex Hull Problem [AS07]

Determine all possible convex decompositions of ρ_T $\rightsquigarrow$ all possible square roots

$$Z = \rho_T^{1/2} R, \quad R \text{ right-unitary.}$$

Theorem

An unital quantum channel T is a convex combination of unitary channels if and only if there is a right-unitary $d^2 \times K$ matrix R $(K \geq d^2)$ such that

$$\mathrm{diag}\left(R^\dagger G_i R\right) = 0 \text{ for all } i = 1, \ldots d^2 - 1,$$
$$G_i := \rho_T^{1/2} \left(\tau_i \otimes \mathbb{1}\right) \rho_T^{1/2}.$$

By Caratheodory's theorem, $K \leq d^4 + 1$ suffices.

Separation Witnesses

Idea: use separating hyperplanes between a given class of quantum channels $\mathcal{S}$ and channels not in $\mathcal{S}$.

Theorem (Hahn-Banach)

Let $\mathcal{S}$ be a bounded, closed, convex subset of the set of all quantum channels, and let T be a quantum channel not in $\mathcal{S}$. Then there exists a Hermitian operator W such that (in the Jamiolkowski representation)

$$\mathrm{tr}\left[W \rho_T\right] < 0, \quad \textit{but} \quad \mathrm{tr}\left[W \sigma\right] \geq 0 \ \forall \sigma \in \mathcal{S}.$$

W is called a separation witness.

Separation Witnesses (continued)

Apply this to the convex hull of unitary channels for certain classes of separation witnesses. Need bounds on $\mathrm{tr}\left[W \rho_T\right]$ for all unitary channels T.

- $W = (A \otimes \mathbb{1})\,\mathbb{F}\left(A^\dagger \otimes \mathbb{1}\right)$, $A \in \mathcal{B}(\mathcal{H})$ arbitrary (flip operator $\mathbb{F} = \sum_{i,j=1}^{d} |ij\rangle\langle ji|$) $\rightsquigarrow$

  $$\left\{ \begin{array}{ll} -2\sum_{i=1}^{d/2} \sigma_{2i-1}\sigma_{2i}, & d \text{ even} \\ -2\sum_{i=1}^{d-1/2} \sigma_{2i-1}\sigma_{2i} + \sigma_d^2, & d \text{ odd} \end{array} \right\} \leq d \cdot \mathrm{tr}\left[W\,\rho_T\right] \leq \|A\|_2^2$$

  with $\sigma_1 \geq \cdots \geq \sigma_d$ the singular values of A.

- $W = \alpha\mathbb{F} + \beta\,|\Omega\rangle\langle\Omega| \rightsquigarrow$ maximize $|\mathrm{tr}\,U|^2$ for fixed $\mathrm{tr}\left[U\overline{U}\right]$ and U unitary, see next slide (covariant channels)

Covariant Channels [VW01]

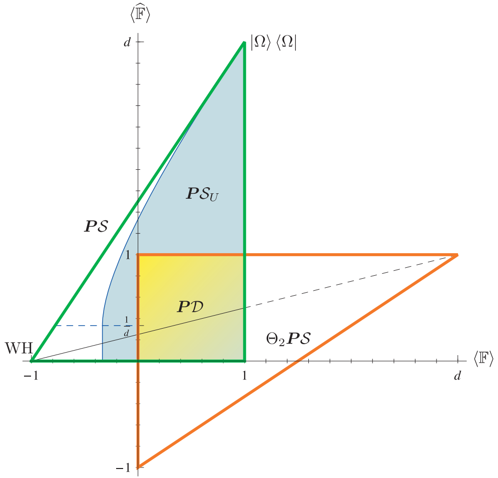

Figure: Covariant channels: $\rho_T = (O \otimes O)\,\rho_T\,(O \otimes O)^\dagger$

Negativity as Distance Measure

Every unital channel T is an affine-linear combination of unitary channels $\rightsquigarrow$ define a negativity canonically as

$$\mathcal{N}(\rho_T) := \inf\left\{\alpha \,:\, \rho_T = (1+\alpha)\,\rho^+ - \alpha\,\rho^-, \alpha \geq 0,\ \rho^\pm \in \mathcal{S}_U\right\}.$$

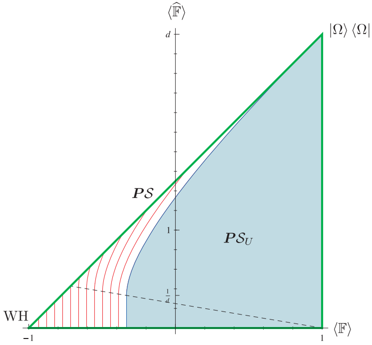

Figure: Negativity

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