

Quantum Boltzmann equation for spin-dependent reactions in the kinetic regime

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2015 J. Phys. A: Math. Theor. 48 095204

(<http://iopscience.iop.org/1751-8121/48/9/095204>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.187.254.46

This content was downloaded on 11/02/2015 at 07:21

Please note that [terms and conditions apply](#).

Quantum Boltzmann equation for spin-dependent reactions in the kinetic regime

Martin L R Fürst^{1,2}, Markus Kotulla³, Christian B Mendl² and Herbert Spohn^{2,3}

¹Excellence Cluster Universe, Boltzmannstraße 2, Technische Universität München, D-85747 Garching bei München, Germany

²Zentrum Mathematik, Boltzmannstraße 3, Technische Universität München, D-85747 Garching bei München, Germany

³Physik Department, James-Franck-Straße 1, Technische Universität München, D-85747 Garching bei München, Germany

E-mail: mfuerst@ma.tum.de, markus.kotulla@tum.de, mendl@ma.tum.de and spohn@ma.tum.de

Received 12 November 2014, revised 28 December 2014

Accepted for publication 7 January 2015

Published 10 February 2015



CrossMark

Abstract

We derive and analyze an effective quantum Boltzmann equation in the kinetic regime for the interactions of four distinguishable types of fermionic spin- $\frac{1}{2}$ particles, starting from a general quantum field Hamiltonian. Each particle type is described by a time-dependent, 2×2 spin-density ('Wigner') matrix. We show that density and energy conservation laws as well as the H-theorem hold, and enumerate additional conservation laws depending on the interaction. The conserved quantities characterize the $t \rightarrow \infty$ thermal (Fermi–Dirac) equilibrium state. We illustrate the approach to equilibrium by numerical simulations in the isotropic three-dimensional setting.

Keywords: quantum Boltzmann equation, weak interaction, kinetic regime, spin-dependent quantum processes

(Some figures may appear in colour only in the online journal)

1. Introduction

Spin-dependent interactions on the quantum level give rise to a wide range of phenomena, for example, the quantum coherence preserving charge and energy transfer during photosynthesis [1, 2], avian navigation of birds [3, 4] or quantum transport in condensed matter physics [5–7], and are even investigated in astrophysics [8]. The dynamics can typically be

modeled by a Hamiltonian on the level of quantum field theory, but solving the resulting equations is often difficult in practice, such that effective approximations are desirable.

Here, we consider the limit of a weak potential interaction term λV with $\lambda \ll 1$ in a general quantum field Hamiltonian (see section 2), and systematically derive and analyze an effective quantum Boltzmann equation in the kinetic regime (section 3) which describes the interactions of four fermionic spin- $\frac{1}{2}$ particles. In particular, we prove the H-theorem and discuss the conservation laws depending on the interaction (see section 4), and present a detailed analysis of the relation between the conserved quantities and the $t \rightarrow \infty$ thermal equilibrium state (see section 5). Finally, we illustrate the approach to equilibrium by numerical simulations in the isotropic three-dimensional setting (sections 6 and 7). The main differences compared to previous work [9, 10] are the four particle types and the continuous domain for the momentum.

2. Multi-component field Hamiltonian

We consider fermionic spin- $\frac{1}{2}$ fields in a d -dimensional box $U = [-\ell, \ell]^d$, with creation and annihilation operators $a_\sigma^\alpha(x)^*$, $a_\sigma^\alpha(x)$, where $\sigma \in \{\uparrow, \downarrow\}$ denotes the spin and $\alpha \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ the particle type. The operators for the same type obey the fermionic anticommutator relations

$$\begin{aligned} \{a_\sigma^\alpha(x)^*, a_\tau^\alpha(y)\} &= \delta(x-y) \delta_{\sigma\tau}, \\ \{a_\sigma^\alpha(x), a_\tau^\alpha(y)\} &= 0, \\ \{a_\sigma^\alpha(x)^*, a_\tau^\alpha(y)^*\} &= 0 \end{aligned} \quad (1)$$

with $\{A, B\} = AB + BA$. The operators for differing particles commute, i.e.,

$$[a_\sigma^\alpha(x)^*, a_\tau^\beta(y)] = 0, \quad \alpha \neq \beta. \quad (2)$$

with the commutator $[A, B] = AB - BA$.

Formally, the underlying one-particle Hilbert space for each particle type is $L^2(U, \mathbb{C}^2)$, and the full Hilbert space is the tensor product of the Fock spaces for the individual particle types.

Our field Hamiltonian is given by

$$H = H_0 + \lambda H_1 \quad (3)$$

with $0 < \lambda \ll 1$ and

$$H_0 = \int_U dx a(x)^* \cdot \hat{\omega}(x) \cdot a(x) \quad (4)$$

as well as

$$\begin{aligned} H_1 = \int_{U^4} dx_{1234} & \left[\left(a(x_1)^* \cdot \mathbf{V}^{\mathbf{ab}} \cdot a(x_2) \right) \left(a(x_3)^* \cdot \mathbf{V}^{\mathbf{cd}} \cdot a(x_4) \right) \right. \\ & \left. + \left(a(x_1)^* \cdot \mathbf{V}^{\mathbf{ad}} \cdot a(x_2) \right) \left(a(x_3)^* \cdot \mathbf{V}^{\mathbf{cb}} \cdot a(x_4) \right) \right] + \text{h.c.} \end{aligned} \quad (5)$$

Here, the $a(x)$ are operator-valued vectors

$$a(x) = \left(a_\uparrow^{\mathbf{a}}(x), a_\downarrow^{\mathbf{a}}(x), a_\uparrow^{\mathbf{b}}(x), \dots, a_\downarrow^{\mathbf{d}}(x) \right) \quad (6)$$

and $\mathbf{V}^{\alpha\beta}$ are 8×8 matrices to be specified below ($\alpha, \beta \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$). $\hat{\omega}(x)$ in equation (4) is the Fourier transform of the dispersion relation.

Historically, Enrico Fermi derived [11] an explanation of the β decay using a Hamiltonian of the form (3). Fermi's four-fermion theory could also predict the weak interaction remarkably well. In this work, our aim is a generalization to spin-dependent interactions.

We will use the following convention for the Fourier transform (corresponding to the finite volume U)

$$\hat{f}(p) = \int_U dx f(x) e^{-ip \cdot x} \quad (7)$$

and the inverse Fourier transform

$$f(x) = \frac{1}{|U|} \sum_{p \in \hat{U}} \hat{f}(p) e^{ip \cdot x}, \quad (8)$$

with $L = 2\ell$ and $\hat{U} = \frac{2\pi}{L} \mathbb{Z}^d$. $|U| = L^d$ denotes the volume of the box. Accordingly, the anticommutator relations in momentum space read

$$\begin{aligned} \{\hat{a}_\sigma^\alpha(p)^*, \hat{a}_\tau^\alpha(p')\} &= |U| \delta(p - p') \delta_{\sigma\tau}, \\ \{\hat{a}_\sigma^\alpha(p), \hat{a}_\tau^\alpha(p')\} &= 0, \\ \{\hat{a}_\sigma^\alpha(p)^*, \hat{a}_\tau^\alpha(p')^*\} &= 0. \end{aligned} \quad (9)$$

The kinetic part of the Hamiltonian in momentum space reads

$$H_0 = \frac{1}{|U|} \sum_{p \in \hat{U}} \hat{a}(p)^* \cdot \omega(p) \cdot \hat{a}(p). \quad (10)$$

Here, the dispersion relations

$$\omega^\alpha(p) = \frac{|p|^2}{2m^\alpha} \quad (11)$$

for each particle α with mass m^α are summarized in the 8×8 diagonal matrix

$$\omega(p) = \text{diag}[\omega^a(p), \omega^b(p), \omega^c(p), \omega^d(p)] \otimes \mathbb{1}_{2 \times 2}. \quad (12)$$

The 2×2 identity matrices appear in spin space since the kinetic energy is independent of spin. The *finite* box $U \subset \mathbb{R}^d$ ensures that the Fourier transform of the dispersion relation in equation (11) is well-defined.

The interaction part of the Hamiltonian in momentum space is given by

$$H_1 = H_1^{\text{abcd}} + H_1^{\text{adcb}} + \text{h.c.} \quad (13)$$

with

$$\left(H_1^{\alpha\beta\gamma\delta}\right)^* = H_1^{\delta\gamma\beta\alpha} \quad (14)$$

and

$$H_1^{\alpha\beta\gamma\delta} = \frac{1}{|U|^4} \sum_{p_{1234}} \delta(\underline{p}) \left(\hat{a}(p_1)^* \cdot \mathbf{V}^{\alpha\beta} \cdot \hat{a}(p_2) \right) \left(\hat{a}(p_3)^* \cdot \mathbf{V}^{\gamma\delta} \cdot \hat{a}(p_4) \right). \quad (15)$$

Here $\underline{p} = p_1 - p_2 + p_3 - p_4$ is the momentum difference, $\sum_{p_{1234}} = \sum_{p_1, p_2, p_3, p_4}$, and we have introduced the 8×8 matrices

$$\mathbf{V}^{ab} = \begin{pmatrix} 0 & V^{ab} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V}^{cd} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V^{cd} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (16)$$

$$\mathbf{V}^{ad} = \begin{pmatrix} 0 & 0 & 0 & V^{ad} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V}^{cb} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V^{cb} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (17)$$

The Hamiltonian should model the interactions

$$\begin{array}{l} a \longleftarrow b, \quad a \longrightarrow b \\ c \longleftarrow d, \quad c \longrightarrow d \end{array} \quad (18)$$

and

$$\begin{array}{l} a \begin{array}{c} \swarrow \\ \searrow \end{array} b, \quad a \begin{array}{c} \searrow \\ \swarrow \end{array} b \\ c \begin{array}{c} \swarrow \\ \searrow \end{array} d, \quad c \begin{array}{c} \searrow \\ \swarrow \end{array} d \end{array} \quad (19)$$

To quantify the (possibly spin-dependent) strength of the interactions, we introduce the 2×2 real-valued ‘interaction matrices’ V^{ab} , V^{cd} , V^{ad} and V^{cb} in momentum space. They model the interactions

$$V^{\alpha\beta} : \alpha \longleftarrow \beta, \quad (V^{\alpha\beta})^* = V^{\beta\alpha} : \alpha \longrightarrow \beta \quad (20)$$

with $\alpha, \beta \in \{a, b, c, d\}$. For simplicity, we assume that these matrices are constant (independent of p). Note that they permit spin dependent reactions like

$$(a_{\uparrow}, c_{\downarrow}) \longrightarrow (b_{\downarrow}, d_{\downarrow}). \quad (21)$$

The system respects conservation of energy and overall particle number. We denote the particle number operator for field α by

$$\hat{N}^{\alpha} = \sum_{p \in \hat{U}, \sigma \in \{\uparrow, \downarrow\}} \hat{a}_{\sigma}^{\alpha}(p)^* \hat{a}_{\sigma}^{\alpha}(p) \quad (22)$$

and thus the total particle number operator reads

$$\hat{N} = \sum_{\alpha \in \{a, b, c, d\}} \hat{N}^{\alpha}. \quad (23)$$

It satisfies the relation $[H, \hat{N}] = 0$, as required. Certain sums of two particles are also conserved

$$\begin{array}{l} [H, \hat{N}^a + \hat{N}^b] = 0, \quad [H, \hat{N}^c + \hat{N}^d] = 0, \\ [H, \hat{N}^a + \hat{N}^d] = 0, \quad [H, \hat{N}^c + \hat{N}^b] = 0, \end{array} \quad (24)$$

since the Hamiltonian only includes the processes in equations (18) and (19). Concerning $\hat{N}^a + \hat{N}^b$, for example, the creation of a involves a simultaneous annihilation of b according to the Hamiltonian structure (15) and hence the sum $\hat{N}^a + \hat{N}^b$ remains constant. Note that not all combinations of two particle types are conserved, e.g.,

$$[H, \hat{N}^a + \hat{N}^c] \neq 0, \quad [H, \hat{N}^b + \hat{N}^d] \neq 0. \quad (25)$$

3. Boltzmann kinetic equation

We will derive the kinetic Boltzmann equation in appendix A. The central object are the two-point functions $W^\alpha(p, t)$, defined by the relation

$$\langle \hat{a}_\sigma^\alpha(p, t)^* \hat{a}_\tau^\beta(p', t) \rangle = |U| \delta_{\alpha\beta} \delta(p - p') W_{\sigma\tau}^\alpha(p, t) \quad (26)$$

for all particle types $\alpha, \beta \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. We collect the 2×2 positive semidefinite (spin density) Wigner states $W^\alpha(p, t)$ in a 8×8 block-diagonal matrix

$$W_1 = \text{diag} \left[W_1^{\mathbf{a}}, W_1^{\mathbf{b}}, W_1^{\mathbf{c}}, W_1^{\mathbf{d}} \right], \quad (27)$$

where we have used the notation $W_1 = W(p_1, t)$. The resulting Boltzmann equation reads

$$\frac{\partial}{\partial t} W(p, t) = C[W](p, t) \quad (28)$$

with the collision operator consisting of a conservative and dissipative part

$$C[W](p, t) = C_{\text{cons}}[W](p, t) + C_{\text{diss}}[W](p, t). \quad (29)$$

C_{cons} and C_{diss} both preserve the block-diagonal structure.

The conservative collision operator C_{cons} is the Vlasov-type operator

$$C_{\text{cons}}[W](p, t) = -i[H_{\text{eff}}(p, t), W(p, t)], \quad (30)$$

where the effective Hamiltonian $H_{\text{eff}}(p, t)$ is a 8×8 block-diagonal matrix which itself depends on W :

$$H_{\text{eff},1} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^{3d}} dp_{234} \delta(\underline{p}) \mathcal{P}(\underline{\omega}^{-1}) h_{\text{eff}}[W]_{234}. \quad (31)$$

The energy differences are defined as

$$\underline{\omega} = \text{diag} \left[\omega^{\text{abcd}}, \omega^{\text{badc}}, \omega^{\text{cdab}}, \omega^{\text{dcba}} \right] \otimes \mathbb{1}_{2 \times 2} \quad (32)$$

with $\omega^{\alpha\beta\gamma\delta} = \omega^\alpha(p_1) - \omega^\beta(p_2) + \omega^\gamma(p_3) - \omega^\delta(p_4)$. In equation (31) we have used the shorthand notation $dp_{234} = dp_2 dp_3 dp_4$. Note that the expression $\mathcal{P}(\underline{\omega}^{-1})$ is a diagonal *matrix* of principal values. The index 234 means that the block-diagonal matrix $h_{\text{eff}}[W]$ depends on p_2, p_3 , and p_4 . It is given by

$$\begin{aligned} h_{\text{eff}}[W]_{234} = & -V^\dagger W_2 V^\dagger \tilde{W}_3 V^\dagger W_4 V^\dagger - V^\dagger W_4 V^\dagger - V^\dagger W_4 V^\dagger \tilde{W}_3 V^\dagger W_2 V^\dagger \\ & - V^\dagger \tilde{W}_2 V^\dagger W_3 V^\dagger \tilde{W}_4 V^\dagger - V^\dagger \tilde{W}_4 V^\dagger - V^\dagger W_3 V^\dagger \tilde{W}_2 V^\dagger \\ & - V^\dagger W_2 V^\dagger \cdot \text{tr} \left[Y \tilde{W}_3 V^\dagger W_4 V^\dagger Y \right] \\ & - V^\dagger \tilde{W}_2 V^\dagger \cdot \text{tr} \left[Y W_3 V^\dagger \tilde{W}_4 V^\dagger Y \right] \\ & - V^\dagger W_4 V^\dagger \cdot \text{tr} \left[Y \tilde{W}_3 V^\dagger W_2 V^\dagger Y \right] \\ & - V^\dagger \tilde{W}_4 V^\dagger \cdot \text{tr} \left[Y W_3 V^\dagger \tilde{W}_2 V^\dagger Y \right], \end{aligned} \quad (33)$$

using the notation $\tilde{W}_i = \mathbb{1}_{8 \times 8} - W_i$. The tr operator appearing in equation (33) acts separately on each (2×2) diagonal block, i.e.,

$$\text{tr} [W] = \sum_{\alpha \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}} E^\alpha \text{tr} [E^\alpha W], \quad E^\alpha = |e_\alpha\rangle \langle e_\alpha| \otimes \mathbb{1}_{2 \times 2} \quad (34)$$

with e_α enumerating the standard basis of \mathbb{R}^4 . The operator Y appearing in equation (33) switches the particle types $(\mathbf{a}, \mathbf{b}) \leftrightarrow (\mathbf{c}, \mathbf{d})$ and is defined as

$$Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2 \times 2}. \quad (35)$$

The 8×8 interaction matrices read

$$V^= = \begin{pmatrix} 0 & V^{ab} & 0 & 0 \\ V^{ba} & 0 & 0 & 0 \\ 0 & 0 & 0 & V^{cd} \\ 0 & 0 & V^{dc} & 0 \end{pmatrix} \quad (36)$$

and

$$V^x = \begin{pmatrix} 0 & 0 & 0 & V^{ad} \\ 0 & 0 & V^{bc} & 0 \\ 0 & V^{cb} & 0 & 0 \\ V^{da} & 0 & 0 & 0 \end{pmatrix}, \quad (37)$$

where always $V^{\beta\alpha} = (V^{\alpha\beta})^*$. The superscripts of $V^=$ and V^x refer to the arrows in equations (18) and (19).

It turns out that the interaction matrices enter the collision operator only via the following 4×4 matrix

$$\mathcal{V} = (V^{ab} \otimes V^{cd}) + (V^{ad} \otimes V^{cb})T, \quad (38)$$

with

$$T = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \in \mathbb{R}^{4 \times 4} \quad (39)$$

an operator which interchanges tensor components (represented in the standard basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$). For example, the (\mathbf{a}, \mathbf{a}) -component (first 2×2 block) of the integrand $h_{\text{eff}}[W]_{234}$ can be represented as

$$\begin{aligned} \langle \sigma | (h_{\text{eff}}[W]_{234})^{\mathbf{a}} | \tau \rangle &= \text{tr} \left[(|\tau\rangle\langle\sigma| \otimes \tilde{W}_3^{\mathbf{c}}) \mathcal{V} (W_2^{\mathbf{b}} \otimes W_4^{\mathbf{d}}) \mathcal{V}^* \right. \\ &\quad \left. + (|\tau\rangle\langle\sigma| \otimes W_3^{\mathbf{c}}) \mathcal{V} (W_2^{\mathbf{b}} \otimes \tilde{W}_4^{\mathbf{d}}) \mathcal{V}^* \right] \end{aligned} \quad (40)$$

with the notation $\tilde{W}_i^\alpha = \mathbb{1}_{2 \times 2} - W_i^\alpha$. Note that $h_{\text{eff}}[W]_{234}$ is invariant under $W_i^\alpha \leftrightarrow \tilde{W}_i^\alpha$, and formally similar to equation (44). The other components arise from the (\mathbf{a}, \mathbf{a}) -component by permutations of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, as for the dissipative operator.

The dissipative part of the collision operator is

$$C_{\text{diss}}[W]_{\mathbf{l}} = \frac{\pi}{(2\pi)^3} \int_{\mathbb{R}^{3d}} d\mathbf{p}_{234} \delta(\underline{p}) \delta(\underline{\omega}) \cdot (\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234}), \quad (41)$$

where the index 1234 means that the block-diagonal matrices $\mathcal{A}_{\text{quad}}[W]$ and $\mathcal{A}_{\text{tr}}[W]$ depend on p_1, p_2, p_3 , and p_4 . They are given by

$$\begin{aligned} \mathcal{A}_{\text{quad}}[W]_{1234} = & + \tilde{W}_1 V^= W_2 V^\times \tilde{W}_3 V^= W_4 V^\times - W_1 V^= \tilde{W}_2 V^\times W_3 V^= \tilde{W}_4 V^\times \\ & + \tilde{W}_1 V^\times W_4 V^= \tilde{W}_3 V^\times W_2 V^= - W_1 V^\times \tilde{W}_4 V^= W_3 V^\times \tilde{W}_2 V^= + \text{h.c.} \end{aligned} \quad (42)$$

and

$$\begin{aligned} \mathcal{A}_{\text{tr}}[W]_{1234} = & + (\tilde{W}_1 V^= W_2 V^= + \text{h.c.}) \cdot \text{tr} [Y \tilde{W}_3 V^= W_4 V^= Y] \\ & - (W_1 V^= \tilde{W}_2 V^= + \text{h.c.}) \cdot \text{tr} [Y W_3 V^= \tilde{W}_4 V^= Y] \\ & + (\tilde{W}_1 V^\times W_4 V^\times + \text{h.c.}) \cdot \text{tr} [Y \tilde{W}_3 V^\times W_2 V^\times Y] \\ & - (W_1 V^\times \tilde{W}_4 V^\times + \text{h.c.}) \cdot \text{tr} [Y W_3 V^\times \tilde{W}_2 V^\times Y]. \end{aligned} \quad (43)$$

If any of the two matrices $V^=$ or V^\times is zero, then $\mathcal{A}_{\text{quad}}[W] = 0$, and the first two or last two terms of $\mathcal{A}_{\text{tr}}[W]$ disappear. Note that $W_i^\alpha \leftrightarrow \tilde{W}_i^\alpha$ effectively switches signs in equations (42) and (43), and that the respective last two terms equal the first two after switching $\mathbf{b} \leftrightarrow \mathbf{d}$ and $2 \leftrightarrow 4$.

Performing the matrix multiplications in equations (42) and (43) shows that Wigner matrices with particle types α and β are always coupled by the respective $V^{\alpha\beta}$ matrix, e.g., $\tilde{W}_i^\alpha V^{\alpha\beta} W_j^\beta$. Additionally, the (\mathbf{b}, \mathbf{b}) -component arises from the (\mathbf{a}, \mathbf{a}) -component by permuting $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{c} \leftrightarrow \mathbf{d}$. Analogously, the (\mathbf{c}, \mathbf{c}) -component arises from (\mathbf{a}, \mathbf{a}) by permuting $\mathbf{a} \leftrightarrow \mathbf{c}$, $\mathbf{b} \leftrightarrow \mathbf{d}$, and the (\mathbf{d}, \mathbf{d}) -component arises from the (\mathbf{a}, \mathbf{a}) -component by permuting $\mathbf{a} \leftrightarrow \mathbf{d}$, $\mathbf{b} \leftrightarrow \mathbf{c}$.

Algebraic reformulation of the (\mathbf{a}, \mathbf{a}) -component of the integrand $\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234}$ results in

$$\begin{aligned} \langle \sigma | (\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234})^{\mathbf{a}} | \tau \rangle \\ = \text{tr} \left[\left(\{ \tilde{W}_1^{\mathbf{a}}, |\tau\rangle \langle \sigma| \} \otimes \tilde{W}_3^{\mathbf{c}} \right) \mathcal{V} \left(W_2^{\mathbf{b}} \otimes W_4^{\mathbf{d}} \right) \mathcal{V}^* \right. \\ \left. - \left(\{ W_1^{\mathbf{a}}, |\tau\rangle \langle \sigma| \} \otimes W_3^{\mathbf{c}} \right) \mathcal{V} \left(\tilde{W}_2^{\mathbf{b}} \otimes \tilde{W}_4^{\mathbf{d}} \right) \mathcal{V}^* \right] \end{aligned} \quad (44)$$

for all spin components σ, τ , where $\{ \cdot, \cdot \}$ denotes the anticommutator. Equivalent expressions give the \mathbf{b}, \mathbf{c} and \mathbf{d} components after appropriate interchanges of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as above, with the anticommutator acting on $W^{\mathbf{b}}, W^{\mathbf{c}}$ and $W^{\mathbf{d}}$, respectively. For example, after a short reformulation

$$\begin{aligned} \langle \sigma | (\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234})^{\mathbf{b}} | \tau \rangle \\ = \text{tr} \left[- \left(\tilde{W}_2^{\mathbf{a}} \otimes \tilde{W}_4^{\mathbf{c}} \right) \mathcal{V} \left(\{ W_1^{\mathbf{b}}, |\tau\rangle \langle \sigma| \} \otimes W_3^{\mathbf{d}} \right) \mathcal{V}^* \right. \\ \left. + \left(W_2^{\mathbf{a}} \otimes W_4^{\mathbf{c}} \right) \mathcal{V} \left(\{ \tilde{W}_1^{\mathbf{b}}, |\tau\rangle \langle \sigma| \} \otimes \tilde{W}_3^{\mathbf{d}} \right) \mathcal{V}^* \right]. \end{aligned} \quad (45)$$

4. General properties of the kinetic equation

The kinetic equation inherits density and energy conservation laws of the Hamiltonian system, as shown below, and the H-theorem holds. Specifically for the multi-component system, there emerge additional conserved quantities depending on the special structure of the $V^{\alpha\beta}$ matrices. In this context, the evolution dynamics is invariant under unitary rotations with fixed unitary $U^\alpha \in \text{SU}(2)$ (separately for each block and independent of p and t), i.e.,

simultaneously

$$\begin{aligned} W^\alpha &\rightarrow U^\alpha W^\alpha (U^\alpha)^*, \\ V^{\alpha\beta} &\rightarrow U^\alpha V^{\alpha\beta} (U^\beta)^*, \\ \mathcal{V} &\rightarrow (U^a \otimes U^c) \mathcal{V} (U^b \otimes U^d)^*, \end{aligned} \quad (46)$$

which can be seen from the representation in equation (44).

4.1. Density conservation

We define the spin density matrix of particle type α as

$$\rho^\alpha(t) = \int_{\mathbb{R}^d} dp W^\alpha(p, t), \quad (47)$$

and the total spin density matrix as

$$\rho(t) = \sum_{\alpha \in \{a,b,c,d\}} \rho^\alpha(t). \quad (48)$$

The analogue of the particle conservation $[H, N] = 0$ on the kinetic level reads

$$\frac{d}{dt} \text{tr} [\rho(t)] = \frac{d}{dt} \int_{\mathbb{R}^d} dp \text{tr} [W(p, t)] = \int_{\mathbb{R}^d} dp \text{tr} [C[W](p, t)] = 0. \quad (49)$$

Even more strongly, according to equation (24) it should hold that

$$\frac{d}{dt} \text{tr} [\rho^\alpha(t) + \rho^\beta(t)] = \int_{\mathbb{R}^d} dp \text{tr}^{\alpha\beta} [C[W](p, t)] = 0 \quad (50)$$

for $\alpha\beta = ab, cd, ad$ or cb . The trace $\text{tr}^{\alpha\beta}$ is understood to act on the blocks α and β only, i.e.,

$$\text{tr}^{\alpha\beta} [W(p, t)] = \text{tr} [W^\alpha(p, t)] + \text{tr} [W^\beta(p, t)]. \quad (51)$$

Relation (50) holds since the integrand of the dissipative C_{diss} vanishes after appropriate interchange of 1, 2, 3, 4: note that

$$\sum_{\sigma \in \{\uparrow, \downarrow\}} \{W_i^\alpha, |\sigma\rangle\langle\sigma|\} = 2W_i^\alpha, \quad (52)$$

such that for $\alpha\beta = ab$, say, the traces of the a-component in equation (44) and b-component in equation (45) (with $(1, 3) \leftrightarrow (2, 4)$) cancel out. The conservative collision operator C_{cons} inserted into (50) vanishes immediately since C_{cons} is a commutator.

Note that taking the trace is indeed required in equation (49), i.e., the individual spin components are not conserved in general.

4.2. Momentum conservation

Momentum conservation

$$\frac{d}{dt} \int_{\mathbb{R}^d} dp p \text{tr} [W(p, t)] = \int_{\mathbb{R}^d} dp p \text{tr} [C[W](p, t)] = 0 \quad (53)$$

with the dispersion matrix follows from the factor $p \delta(p)$ in the integrand after appropriate interchanges $1 \leftrightarrow 3$, $2 \leftrightarrow 4$ and $(1, 3) \leftrightarrow (2, 4)$. Isotropic states always have zero average momentum.

4.3. Energy conservation

Energy conservation is represented by the equation

$$\frac{d}{dt} \int_{\mathbb{R}^d} dp \operatorname{tr}[\omega(p) \cdot W(p, t)] = \int_{\mathbb{R}^d} dp \operatorname{tr}[\omega(p) \cdot C[W(p, t)]] = 0 \quad (54)$$

with the dispersion matrix $\omega(p)$ defined in equation (12). The term inside the trace is a 8×8 matrix. Similar to the momentum conservation, equation (54) follows from the factor $(\omega_1^a - \omega_2^b + \omega_3^c - \omega_4^d) \delta(\omega_1^a - \omega_2^b + \omega_3^c - \omega_4^d)$ in the integrand after appropriate interchanges $1 \leftrightarrow 3, 2 \leftrightarrow 4$ and $(1, 3) \leftrightarrow (2, 4)$.

4.4. Additional conservation laws depending on the interaction matrices

Taking all conservation laws into account is necessary for computing the asymptotic (thermal) equilibrium state (see section 5 below), and there are additional conservation laws depending on the $V^{\alpha\beta}$ matrices. Since the collision operator can be expressed in terms of the \mathcal{V} matrix in equation (38), it suffices to discuss the structure and zero pattern of the entries of \mathcal{V} , which is to be understood modulo unitary rotations of the form (46). Whenever such rotations lead to a particular pattern as discussed in the following, the respective conservation law holds in this basis.

We will only consider $V^{\alpha\beta}$ matrices with full rank 2, to exclude degenerate cases like $V^{\alpha\beta} = 0$ (as a matrix).

General diagonal $V^{\alpha\beta}$. The \mathcal{V} matrix represented in the standard basis $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$ has the structure

$$\mathcal{V} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad (55)$$

where each star represents an arbitrary number. In this case, the diagonal entries of the total spin remain constant under the time evolution of the Boltzmann equation

$$\frac{d}{dt} \rho_{\sigma\sigma}(t) = \frac{d}{dt} \sum_{\alpha \in \{a,b,c,d\}} \int_{\mathbb{R}^d} dp W_{\sigma\sigma}^{\alpha}(p, t) = 0. \quad (56)$$

To prove this assertion, consider the $\uparrow\uparrow$ entry (the proof for the $\downarrow\downarrow$ entry proceeds analogously). Expanding the representation (44) gives

$$\begin{aligned} & \sum_{\alpha \in \{a,b,c,d\}} \langle \uparrow | (\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234})^{\alpha} | \uparrow \rangle \\ &= \sum_{\sigma, \tau} D(\sigma, \tau) \left(\langle \sigma_1 \sigma_3 | (\tilde{W}_1^a \otimes \tilde{W}_3^c) | \tau_1 \tau_3 \rangle \langle \sigma_2 \sigma_4 | (W_2^b \otimes W_4^d) | \tau_2 \tau_4 \rangle \right. \\ & \quad - \langle \sigma_1 \sigma_3 | (W_1^a \otimes W_3^c) | \tau_1 \tau_3 \rangle \langle \sigma_2 \sigma_4 | (\tilde{W}_2^b \otimes \tilde{W}_4^d) | \tau_2 \tau_4 \rangle \left. \right) \\ & \quad \times \langle \tau_1 \tau_3 | \mathcal{V} | \sigma_2 \sigma_4 \rangle \langle \tau_2 \tau_4 | \mathcal{V}^* | \sigma_1 \sigma_3 \rangle \end{aligned} \quad (57)$$

with

$$D(\sigma, \tau) = \left(\delta_{\sigma_1, \uparrow} + \delta_{\tau_1, \uparrow} - \delta_{\sigma_2, \uparrow} - \delta_{\tau_2, \uparrow} + \delta_{\sigma_3, \uparrow} + \delta_{\tau_3, \uparrow} - \delta_{\sigma_4, \uparrow} - \delta_{\tau_4, \uparrow} \right) \quad (58)$$

and the notation $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4)$. Direct inspection shows that $D(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 0$ or $\langle \tau_1 \tau_3 | \mathcal{V} | \sigma_2 \sigma_4 \rangle \langle \tau_2 \tau_4 | \mathcal{V}^* | \sigma_1 \sigma_3 \rangle = 0$ for all spin combinations, given the zero pattern in equation (55).

There are $5 + d$ independently conserved quantities: the two diagonal entries in equation (56), the densities of $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{d}$ according to equation (50), the momentum and the total energy. The other conserved quantities are redundant; for example, the density of $\mathbf{c} + \mathbf{d}$ can be obtained from the sum of the diagonal entries in equation (56) minus the density of $\mathbf{a} + \mathbf{b}$.

All $V^{\alpha\beta}$ proportional to the identity matrix. This is a special case of (a), relevant for the β decay discussed below, and \mathcal{V} is of the form $\mathcal{V} = c^- \mathbb{1}_{4 \times 4} + c^x T$ with two constants c^- and c^x . In this case \mathcal{V} is invariant under a simultaneous unitary rotation of the Wigner matrices as in equation (46) with $U^a = U^b = U^c = U^d = U \in \text{SU}(2)$, i.e., $(U \otimes U) \mathcal{V} (U \otimes U)^* = \mathcal{V}$. Such a simultaneous rotation sends $\rho(t) \rightarrow U \rho(t) U^*$, and together with equation (56), it follows that the total spin density matrix remains constant in time

$$\frac{d}{dt} \rho(t) = 0. \quad (59)$$

Alternatively, one could prove this assertion starting directly from equation (44), together with the identities $\text{tr}[A \otimes B] = \text{tr}[A] \text{tr}[B]$ and $\text{tr}[\{A, C\} \cdot B] = \text{tr}[A \cdot \{B, C\}]$, which are valid for any matrices A, B and C .

Zero outer frame in \mathcal{V} matrix. We investigate the zero pattern

$$\mathcal{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (60)$$

represented in the standard basis ($|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$) as above. This pattern can emerge from non-diagonal $V^{\alpha\beta}$ interaction matrices with full rank, too. Besides the conservation of the diagonal entries in equation (56), the projection onto the Pauli matrix $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for types $\mathbf{a} + \mathbf{c}$ and $\mathbf{b} + \mathbf{d}$ is also conserved, i.e.,

$$\frac{d}{dt} \sum_{\alpha} \text{tr}[\sigma_z \rho^{\alpha}(t)] = \frac{d}{dt} \sum_{\alpha} \int_{\mathbb{R}^d} dp \text{tr}[\sigma_z W^{\alpha}(p, t)] = 0 \quad (61)$$

with summation over $\alpha \in \{\mathbf{a}, \mathbf{c}\}$ or $\alpha \in \{\mathbf{b}, \mathbf{d}\}$. To prove this statement, first note that

$$\{W_i^{\alpha}, \sigma_z\} = 2 \begin{pmatrix} \langle \uparrow | W_i^{\alpha} | \uparrow \rangle & 0 \\ 0 & -\langle \downarrow | W_i^{\alpha} | \downarrow \rangle \end{pmatrix}. \quad (62)$$

Then we proceed as for diagonal $V^{\alpha\beta}$ above, except that $D(\boldsymbol{\sigma}, \boldsymbol{\tau})$ in equation (57) is replaced by

$$D'(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 2(\langle \sigma_1 | \sigma_z | \tau_1 \rangle + \langle \sigma_3 | \sigma_z | \tau_3 \rangle) \quad (63)$$

for $\alpha \in \{\mathbf{a}, \mathbf{c}\}$. As before, $D'(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 0$ or $\langle \tau_1 \tau_3 | \mathcal{V} | \sigma_2 \sigma_4 \rangle \langle \tau_2 \tau_4 | \mathcal{V}^* | \sigma_1 \sigma_3 \rangle = 0$ for all spin combinations, given the pattern in equation (60).

In summary, there are $6 + d$ independently conserved quantities: the $5 + d$ quantities from case (a) with diagonal $V^{\alpha\beta}$, and the projection onto σ_z in equation (61) with summation over $\alpha \in \{\mathbf{a}, \mathbf{c}\}$. Summation over $\alpha \in \{\mathbf{b}, \mathbf{d}\}$ is redundant due to equation (56).

The independently conserved quantities are summarized in table 1.

4.5. H-theorem

In the following, we prove the H-theorem which states that the entropy is monotonically increasing. We represent each Wigner function by its spectral decomposition

$$W^\alpha(p) = \sum_{\sigma \in \{\uparrow, \downarrow\}} \lambda_\sigma^\alpha(p) P_\sigma^\alpha(p) \quad (64)$$

for $\alpha \in \{a, b, c, d\}$, where $0 \leq \lambda_\sigma^\alpha(p) \leq 1$ are the eigenvalues and $P_\sigma^\alpha(p) = |\alpha; p, \sigma\rangle\langle\alpha; p, \sigma|$ an orthogonal eigenprojector.

The entropy production is given by

$$\sigma[W] = \frac{d}{dt} S[W] = - \int_{\mathbb{R}^3} dp_1 \operatorname{tr} \left[(\log W_1 - \log \tilde{W}_1) C[W]_1 \right]. \quad (65)$$

In the following, we will use the shorthand notation $\lambda_j = \lambda_{\sigma_j}^{(j)}(p_j)$, $P_j = P_{\sigma_j}^{(j)}(p_j) = |j\rangle\langle j|$, and $\sum_\sigma = \sum_{\sigma_1, \sigma_2, \sigma_3, \sigma_4}$. For example, $\lambda_2 = \lambda_{\sigma_2}^b(p_2)$. Inserting the spectral decomposition (64) and the integrand representation (44) of the dissipative collision operator into equation (65), the contribution of the (a, a)-component (first 2×2 block) to the entropy production reads

$$\begin{aligned} \sigma[W]^a &= -\pi \int dp_{1234} \delta(\underline{p}) \delta(\omega_1^a - \omega_2^b + \omega_3^c - \omega_4^d) \\ &\quad \times \operatorname{tr} \left[\left(\{ \tilde{W}_1^a, \log W_1 - \log \tilde{W}_1 \} \otimes \tilde{W}_3^c \right) \mathcal{V} \left(W_2^b \otimes W_4^d \right) \mathcal{V}^* \right. \\ &\quad \left. - \left(\{ W_1^a, \log W_1 - \log \tilde{W}_1 \} \otimes W_3^c \right) \mathcal{V} \left(\tilde{W}_2^b \otimes \tilde{W}_4^d \right) \mathcal{V}^* \right] \\ &= 2\pi \int dp_{1234} \delta(\underline{p}) \delta(\omega_1^a - \omega_2^b + \omega_3^c - \omega_4^d) \\ &\quad \times \sum_{\sigma} (\log \lambda_1 - \log \tilde{\lambda}_1) (\lambda_1 \tilde{\lambda}_2 \lambda_3 \tilde{\lambda}_4 - \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_3 \lambda_4) \\ &\quad \times \langle 13 | \mathcal{V} | 24 \rangle \langle 24 | \mathcal{V}^* | 13 \rangle. \end{aligned} \quad (66)$$

The contribution of the (b, b)-component to the entropy production coincides with equation (66) after permuting $a \leftrightarrow b$, $c \leftrightarrow d$. Together with relabeling the integration variables $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, the contribution of the (b, b)-component has exactly the same form as (66) upon replacing

$$(\log \lambda_1 - \log \tilde{\lambda}_1) \rightarrow -(\log \lambda_2 - \log \tilde{\lambda}_2). \quad (67)$$

Similar reasoning holds for the contributions from the (c, c) and (d, d) components. In summary, the entropy production equals

$$\begin{aligned} \sigma[W] &= 2\pi \int dp_{1234} \delta(\underline{p}) \delta(\omega_1^a - \omega_2^b + \omega_3^c - \omega_4^d) \\ &\quad \times \sum_{\sigma} \log \left(\frac{\lambda_1 \tilde{\lambda}_2 \lambda_3 \tilde{\lambda}_4}{\tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_3 \lambda_4} \right) (\lambda_1 \tilde{\lambda}_2 \lambda_3 \tilde{\lambda}_4 - \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_3 \lambda_4) |\langle 13 | \mathcal{V} | 24 \rangle|^2 \geq 0 \end{aligned} \quad (68)$$

since $(x - y) \log(x/y) \geq 0$.

5. Stationary states

All stationary states have to satisfy $\sigma[W] = 0$, i.e., the entropy production must be zero. To elucidate the set of Wigner functions which adhere to this condition, we define (in the context

Table 1. Independently conserved quantities, depending on the special structure of \mathcal{V} .

Structure of \mathcal{V}	Conserved quantities				
General \mathcal{V}	Momentum (53) and energy (54)	$\text{tr} [\rho^a(t) + \rho^b(t)]$	$\text{tr} [\rho^a(t) + \rho^d(t)]$	$\text{tr} [\rho(t)]$	
\mathcal{V} in equation (55) (general diagonal $V^{\alpha\beta}$)				$\rho_{\uparrow\uparrow}(t)$	$\rho_{\downarrow\downarrow}(t)$
$\mathcal{V} = c^= \mathbb{1}_{4 \times 4} + c^X T$ ($V^{\alpha\beta}$ proportional to identity)				$\rho(t)$	
Zero outer frame in \mathcal{V} matrix (equation (60))				$\rho_{\uparrow\uparrow}(t)$	$\rho_{\downarrow\downarrow}(t) \quad \text{tr} [\sigma_z(\rho^a(t) + \rho^c(t))]$

of the proof of the H-theorem)

$$F(\mathbf{p}, \boldsymbol{\sigma}) = \log \left(\frac{\lambda_1 \tilde{\lambda}_2 \lambda_3 \tilde{\lambda}_4}{\tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_3 \lambda_4} \right) (\lambda_1 \tilde{\lambda}_2 \lambda_3 \tilde{\lambda}_4 - \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_3 \lambda_4) \geq 0 \quad (69)$$

and $\mathcal{V}(\mathbf{p}, \boldsymbol{\sigma}) = \langle 13 | \mathcal{V} | 24 \rangle$, where \mathcal{V} is the 4×4 matrix in equation (38) and we have used the notation $|j\rangle\langle j| = P_{\sigma_j}^{(j)}(p_j)$ from above. It must hold that $F(\mathbf{p}, \boldsymbol{\sigma}) = 0$ or $\mathcal{V}(\mathbf{p}, \boldsymbol{\sigma}) = 0$ (or both) for each configuration of the $\boldsymbol{\sigma}$ variables, according to equation (68). Defining the collision invariants as

$$\Phi_{\sigma}^{\alpha}(p) = \log \left(\frac{\lambda_{\sigma}^{\alpha}(p)}{\tilde{\lambda}_{\sigma}^{\alpha}(p)} \right), \quad (70)$$

then $F(\mathbf{p}, \boldsymbol{\sigma}) = 0$ is equivalent to

$$\Phi_{\sigma_1}^a(p_1) - \Phi_{\sigma_2}^b(p_2) + \Phi_{\sigma_3}^c(p_3) - \Phi_{\sigma_4}^d(p_4) = 0. \quad (71)$$

Based on general arguments [12], one expects that the Wigner functions will equilibrate as $t \rightarrow \infty$, i.e., converge to thermal equilibrium (Fermi–Dirac) distributions

$$W_{\text{eq}}^{\alpha}(p) = \sum_{\sigma \in \{\uparrow, \downarrow\}} \lambda_{\text{eq}, \sigma}^{\alpha}(p) |\alpha; \sigma\rangle \langle \alpha; \sigma| \quad \text{with} \\ \lambda_{\text{eq}, \sigma}^{\alpha}(p) = \left(e^{\beta(\omega^{\alpha}(p) - \mu_{\sigma}^{\alpha})} + 1 \right)^{-1}. \quad (72)$$

Here we have assumed that the orthonormal eigenbasis $|\alpha; \sigma\rangle$ is independent of p (thus $\mathcal{V}(\mathbf{p}, \boldsymbol{\sigma}) = \mathcal{V}(\boldsymbol{\sigma})$), that the average momentum is zero, and that all particle types share the same inverse temperature β . We exclude degenerate cases like $V^{\alpha\beta} = 0$ as a matrix. Inserting the Fermi–Dirac eigenvalues in (72) into (71) and using the energy conservation translates to the *linear* equation

$$F(\mathbf{p}, \boldsymbol{\sigma}) = 0 \quad \Leftrightarrow \quad \mu_{\sigma_1}^a - \mu_{\sigma_2}^b + \mu_{\sigma_3}^c - \mu_{\sigma_4}^d = 0. \quad (73)$$

The remaining task is to determine the chemical potentials μ_{σ}^{α} , inverse temperature β and the basis $|\alpha; \sigma\rangle$ in accordance with the conservation laws, which themselves depend on \mathcal{V} .

For the following, it is convenient to represent the right side of equation (73) for each $\boldsymbol{\sigma}$ combination as 4×4 matrix (denoted by \mathcal{F}) with entries

$$\langle \sigma_1 \sigma_3 | \mathcal{F} | \sigma_2 \sigma_4 \rangle = \mu_{\sigma_1}^a - \mu_{\sigma_2}^b + \mu_{\sigma_3}^c - \mu_{\sigma_4}^d \quad (74)$$

with respect to the standard basis ($|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$). This representation is analogous to the \mathcal{V} matrix.

After changing basis according to equation (46), \mathcal{V} may be represented in the eigenbasis $|\alpha; \sigma\rangle$, and we can without loss of generality assume that $|\alpha; \sigma\rangle$ is the standard basis.

In what follows, we discuss a (non-exhaustive) list of special cases (as for the additional conservation laws in section 4.4).

General \mathcal{V} . We assume that \mathcal{V} exhibits none of the zero patterns below, even after unitary rotations of the form (46). Explicit enumeration using a computer algebra system shows the following: whenever the condition (73) holds for at least nine (pairwise different) configurations of the $\boldsymbol{\sigma}$ variables, then all chemical potentials are necessarily independent of spin

$$\mu_{\sigma}^{\alpha} = \nu^{\alpha} \quad \text{with} \quad \nu^a - \nu^b + \nu^c - \nu^d = 0. \quad (75)$$

In this case $F(\mathbf{p}, \boldsymbol{\sigma}) = 0$ always. According to the first row in table 1, there are four independently conserved quantities (for zero average momentum), and correspondingly four parameters to describe the equilibrium state, namely β , ν^a , ν^b and ν^c (ν^d is fixed by equation (75)). Note that the choice of the basis $|\alpha; \sigma\rangle$ is arbitrary in the present case due to independence of spin.

\mathcal{V} with zero structure in equation (55). This case is equivalent to general diagonal $V^{\alpha\beta}$ matrices. Since $F(\mathbf{p}, \boldsymbol{\sigma}) = 0$ must hold whenever $\mathcal{V}(\boldsymbol{\sigma}) \neq 0$, the required complementary zero pattern for \mathcal{F} reads

$$\mathcal{F} = \begin{pmatrix} 0 & * & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & * \\ * & * & * & 0 \end{pmatrix}. \quad (76)$$

Solving the linear equations (73) corresponding to the zero entries of this matrix leads to the solution

$$\mu_\sigma^\alpha = \nu^\alpha + c \langle \sigma | \sigma_z | \sigma \rangle$$

with a fixed $c \in \mathbb{R}$ and $\nu^a - \nu^b + \nu^c - \nu^d = 0$. There are five independent parameters (in accordance with the five conservation laws in the second row of table 1): the values of β , ν^a , ν^b , ν^c and c .

$\mathcal{V} = c^= \mathbb{1}_{4 \times 4} + c^x T$. This structure results from all $V^{\alpha\beta}$ matrices proportional to the identity matrix, summarized in the third row of table 1. Since $\rho(t)$ remains constant in time, we can diagonalize $\rho(t)$ by a global, constant unitary rotation $U \in \text{SU}(2)$. Thus, without loss of generality one can assume that $\rho(t)$ is diagonal. From here the argumentation proceeds as in the previous case with general diagonal $V^{\alpha\beta}$.

Zero outer frame in \mathcal{V} matrix, equation (60). The complementary zero pattern for \mathcal{F} is

$$\mathcal{F} = \begin{pmatrix} * & * & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & * \\ * & * & * & * \end{pmatrix}. \quad (77)$$

Solving the corresponding system of linear equations according to (73) leads to

$$\mu_\sigma^\alpha = \nu^\alpha + c^\alpha \langle \sigma | \sigma_z | \sigma \rangle \quad (78)$$

with $c^a = c^c$, $c^b = c^d$ and $\nu^a - \nu^b + \nu^c - \nu^d = 0$. The number of independent parameters (β , ν^a , ν^b , ν^c , c^a and c^b) for zero average momentum matches the number of conserved quantities, see last row in table 1.

In practice, we fit β and the additional parameters numerically such that the conserved quantities obtained from the corresponding Fermi–Dirac state match the ones of the initial state. We conjecture that the map from the conserved quantities to the parameters is one to one.

6. Numerical procedure

Concerning the numeric integration for the dissipative collision operator, our goal is to solve the following p_1 -dependent integral numerically:

$$\mathcal{C}_{\text{diss}}[W]_1^\alpha = \pi \int_{\mathbb{R}^9} dp_{234} \delta^3(\underline{p}) \delta(\omega^{\alpha\beta\gamma\delta}) (\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234})^\alpha, \quad (79)$$

where we have used the notation $\underline{p} = p_1 - p_2 + p_3 - p_4$ and $\omega^{\alpha\beta\gamma\delta} = \omega_1^\alpha - \omega_2^\beta + \omega_3^\gamma - \omega_4^\delta$.

We follow the derivation [13, appendix A] to resolve the δ -functions in the collision integral (79) as far as possible and to integrate out the angular parts. Expressed in terms of the energies $\varepsilon_i = |p_i|^2/(2m^i)$ with $m^1 = m^\alpha$, $m^2 = m^\beta$ etc, and using the relation

$$d^3p_i = d\Omega |p_i| m^i d\varepsilon_i, \quad (80)$$

one arrives at the following two-dimensional integral:

$$\begin{aligned} C_{\text{diss}}[W]_1^\alpha &= (2\pi)^3 m^\beta m^\gamma m^\delta \int_{\mathcal{D}(\varepsilon_1)} d\varepsilon_2 d\varepsilon_4 \frac{\min(|p_1|, |p_2|, |p_3|, |p_4|)}{|p_1|} \\ &\times \left(\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234} \right)^\alpha \end{aligned} \quad (81)$$

with the integration domain $\mathcal{D}(\varepsilon_1) = \{ \varepsilon_2, \varepsilon_4 \mid \varepsilon_2 \geq 0, \varepsilon_4 \geq 0, \varepsilon_2 + \varepsilon_4 - \varepsilon_1 \geq 0 \}$ and the relations $|p_i| = \sqrt{2 m^i \varepsilon_i}$ and $\varepsilon_3 = \varepsilon_2 + \varepsilon_4 - \varepsilon_1$. The (unbounded) domain $\mathcal{D}(\varepsilon_1) \subset \mathbb{R}^2$ simply encodes the physical condition that the individual energies must be non-negative. Note that the min-term in equation (81) expressed by the particle energies reads

$$D = \frac{\min(|p_1|, |p_2|, |p_3|, |p_4|)}{|p_1|} = \left(\frac{\min(m^\alpha \varepsilon_1, m^\beta \varepsilon_2, m^\gamma \varepsilon_3, m^\delta \varepsilon_4)}{m^\alpha \varepsilon_1} \right)^{1/2}. \quad (82)$$

The numerical discretization of the integral (81) should preserve the conservation laws, which result from the interchangeability $\varepsilon_1 \leftrightarrow \varepsilon_3$, $\varepsilon_2 \leftrightarrow \varepsilon_4$ and the pairs $\{\varepsilon_1, \varepsilon_3\} \leftrightarrow \{\varepsilon_2, \varepsilon_4\}$. For this reason, we refrain from using Zakharov transformations [13, 14], and instead opt for a uniform grid for the energy variables, as follows. To adopt the symmetries in the numerical discretization, we first rewrite the integral (81):

$$\begin{aligned} &\int_{\mathcal{D}(\varepsilon_1)} d\varepsilon_2 d\varepsilon_4 D \left(\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234} \right)^\alpha \\ &= \int d\varepsilon'_1 d\varepsilon_2 d\varepsilon_3 d\varepsilon_4 \delta(\varepsilon'_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4) \\ &\quad \times \delta(\varepsilon'_1 - \varepsilon_1) D \left(\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234} \right)^\alpha \\ &= 2 \int_{\mathcal{D}'} ds_{13} d\Delta\varepsilon_{13} d\Delta\varepsilon_{24} \delta(s_{13} + \Delta\varepsilon_{13} - \varepsilon_1) \\ &\quad \times D \left(\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234} \right)^\alpha, \end{aligned} \quad (83)$$

where we have used the substitution

$$s_{13} = \frac{1}{2}(\varepsilon'_1 + \varepsilon_3), \quad \Delta\varepsilon_{13} = \frac{1}{2}(\varepsilon'_1 - \varepsilon_3), \quad (84)$$

$$s_{24} = \frac{1}{2}(\varepsilon_2 + \varepsilon_4), \quad \Delta\varepsilon_{24} = \frac{1}{2}(\varepsilon_2 - \varepsilon_4). \quad (85)$$

The domain \mathcal{D}' of the last integral in (83) is defined as

$$\mathcal{D}' = \left\{ s_{13}, \Delta\varepsilon_{13}, \Delta\varepsilon_{24} \mid s_{13} - |\Delta\varepsilon_{13}| \geq 0, s_{13} - |\Delta\varepsilon_{24}| \geq 0 \right\}, \quad (86)$$

corresponding to non-negative energies.

Numerically, we store the Wigner matrices $W^{(\alpha)}(\varepsilon_j)$ discretized on a uniform grid for the energy variable:

$$\varepsilon_j = h j, \quad j = 0, 1, \dots, n-1, \quad (87)$$

with a small grid spacing $0 < h \ll 1$. The same uniform grid is used to approximate the integration with respect to s_{13} , $\Delta\varepsilon_{13}$ and $\Delta\varepsilon_{24}$ in (83), such that the energy values $\varepsilon_3 = s_{13} - \Delta\varepsilon_{13}$, $\varepsilon_2 = s_{13} + \Delta\varepsilon_{24}$ and $\varepsilon_4 = s_{13} - \Delta\varepsilon_{24}$ are always grid points (87). Note that $\varepsilon_1 \leftrightarrow \varepsilon_3$ corresponds to $\Delta\varepsilon_{13} \leftrightarrow -\Delta\varepsilon_{13}$ and likewise for $\Delta\varepsilon_{24}$, and that $\{\varepsilon_1, \varepsilon_3\} \leftrightarrow \{\varepsilon_2, \varepsilon_4\}$ corresponds to $\Delta\varepsilon_{13} \leftrightarrow \Delta\varepsilon_{24}$.

Alternative integration schemes (like the apparent Gauss–Laguerre quadrature rule) were also considered but eventually dismissed in favor of the simple trapezoidal rule on a uniform grid. The main advantages are that the conservation laws are respected by the numerical procedure, and that no interpolation of Wigner matrices is required. The uniform discretization has been suggested before [15]. Unfortunately, the fast algorithm proposed in [15] cannot simply be used here due to the dependence of D in equation (82) on the particle masses.

Different from the one-dimensional case, a mollification procedure as in [9, 10] is not required since the integrals no longer diverge.

Concerning the conservative collision operator C_{cons} , we perform a change of variables to the energies ε_i as for the dissipative operator. The integral (31) for the effective Hamiltonian then reads

$$H_{\text{eff},1}^\alpha = 2(2\pi)^2 m^\beta m^\gamma m^\delta \int d\varepsilon_2 d\varepsilon_3 d\varepsilon_4 \frac{\min(|p_1|, |p_2|, |p_3|, |p_4|)}{|p_1|} \times \mathcal{P}\left((\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)^{-1}\right) h_{\text{eff}}[W]_{234}^\alpha. \quad (88)$$

Analytically, the principal value results in the derivative of the integrand, in the sense that

$$\frac{1}{2} \int_{-h}^h d\varepsilon \mathcal{P}(\varepsilon^{-1}) f(\varepsilon) = h f'(0) + \mathcal{O}(h^3) \quad (89)$$

for any sufficiently smooth function f . In the numerical scheme, we simply omit the grid points for which $\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 = 0$, in order to preserve the conservation laws. The error of this approximation is expected to vanish for grid spacing $h \rightarrow 0$.

To solve the Boltzmann equation, we use the explicit midpoint rule for $C = C_{\text{diss}} + C_{\text{cons}}$ as in [10]. As advantage, this approach exactly preserves the spin and energy conservation laws.

We have implemented the numerical scheme described so far in plain C code, and use the *MathLink* interface to make the numerical procedures conveniently accessible from Mathematica.

7. Simulation results

For the following simulations, we fix an initial Wigner state $W(p, 0)$ with particle masses $m^a = 1$, $m^b = \frac{4}{5}$, $m^c = \frac{1}{5}$ and $m^d = \frac{1}{2}$. Figure 1 illustrates the $W^\alpha(p, 0)$ components in dependence of $|p|$. For reference, the analytical formulas of the initial state are recorded in appendix B. Note that on the quantum field level in (18) and (19), a conservation of masses like $m^a + m^c = m^b + m^d$ is not required.

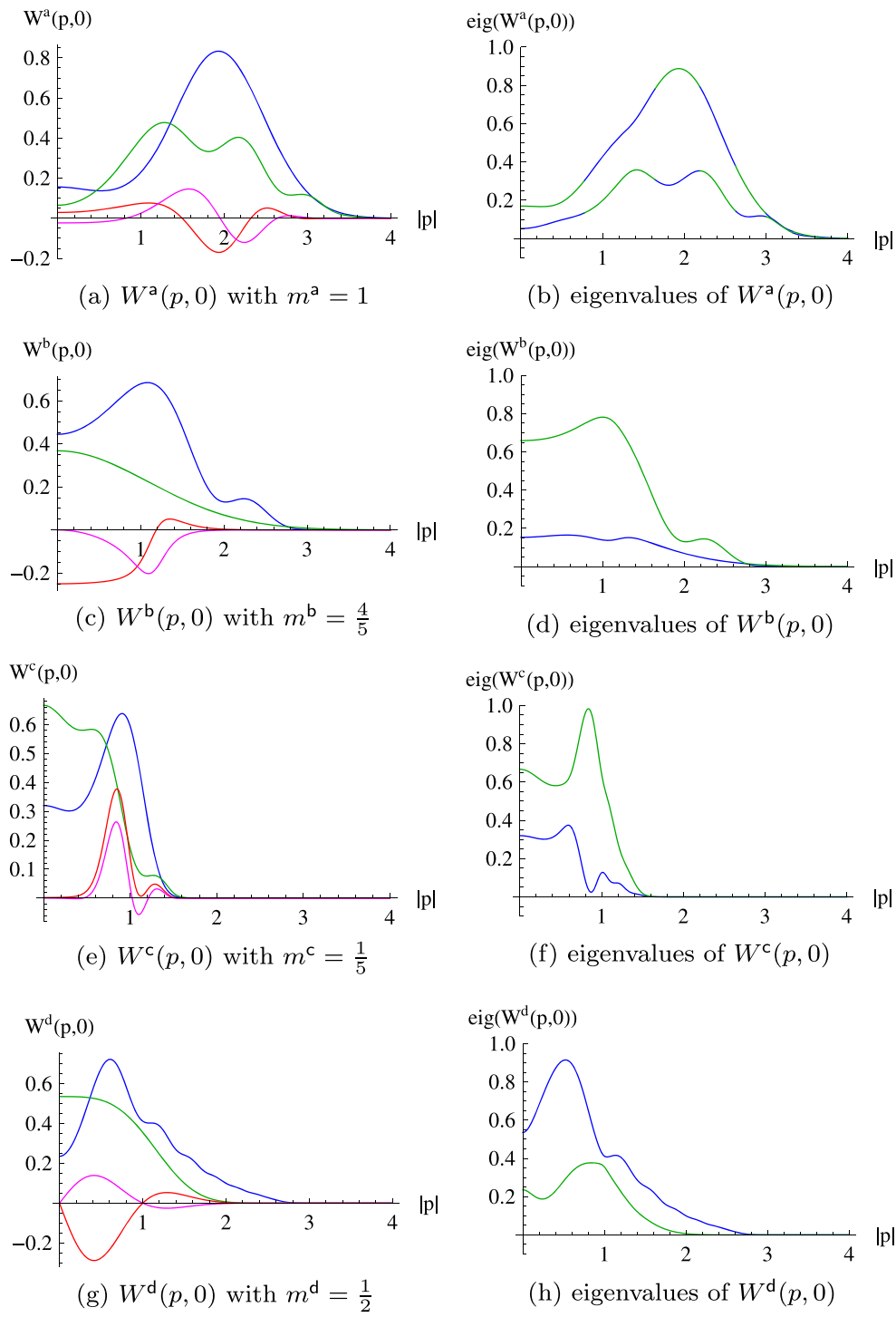


Figure 1. The initial state $W(p,0)$ used for the simulations. Left column: matrix entries: the blue and green (upper) curves show the real diagonal entries, and the red and magenta curves the real and imaginary parts of the off-diagonal $|\uparrow\rangle\langle\downarrow|$ entry, respectively. Right column: corresponding eigenvalues of $W^\alpha(p,0)$.

7.1. Weak interaction: β decay

An application of our framework is the β decay, i.e., the decay of a neutron n into a proton p , an electron e and an antineutrino $\bar{\nu}$. Equivalently, this process can be represented as

$$n + \nu \longleftrightarrow p + e. \quad (90)$$

The interaction part from equation (5) is given by

$$H_1 = \frac{4\pi}{g_w^2} H_\beta, \quad (91)$$

where g_w is the weak coupling factor and

$$H_\beta = \frac{G_F}{\sqrt{2}} \int d^3x \left(\bar{u}^p \gamma^\mu (C_V + C_A \gamma_5) u^n \right) \left(\bar{u}^e \gamma_\mu (1 - \gamma_5) u^\nu \right) \quad (92)$$

the Hamiltonian of the Fermi theory [16]. Einstein summation convention is used for the gamma matrices γ^μ and C_V, C_A are constants satisfying

$$C_A/C_V = -1.255 \pm 0.006. \quad (93)$$

G_F is the Fermi coupling constant. With the relation for the weak coupling constant $\alpha_w = \frac{g_w^2}{4\pi}$, we identify

$$\alpha_w \left(\frac{4\pi}{8m_w^2} \right) = \frac{g_w^2}{8m_w^2} = \frac{G_F}{\sqrt{2}}, \quad (94)$$

where m_w is the mass of the W boson. In our notation of equation (3) the dimensionless weak coupling

$$\lambda = \alpha_w \approx 0.0339. \quad (95)$$

A short calculation shows that the Hamiltonian in equation (92) can be represented in the form of equation (13) by setting

$$V^{np} = (C_V - C_A)\mathbf{1}, \quad V^{\nu e} = \mathbf{1}, \quad (96)$$

$$V^{ne} = \mathbf{1}, \quad V^{\nu p} = 2 C_A \mathbf{1} \quad (97)$$

up to the $g/\sqrt{2}$ prefactor, that is, all interaction matrices are proportional to the identity matrix. Physically, the β decay process is independent of spin.

Figure 2 illustrates asymptotic thermal Fermi–Dirac equilibrium states as determined from the conservation laws. The equilibrium states are represented in the eigenbasis of the total density $\rho(t)$, which remains constant in time according to equation (59). The particle type associations are **a**: neutrons, **b**: protons, **c**: neutrinos and **d**: electrons. The masses are not physically realistic in this model calculation. Our numerical simulation with the interaction matrices in equations (96) and (97) indeed confirms that the Boltzmann equation drives the initial state in figure 1 to these thermal equilibrium states. The entropy convergence is visualized in figure 3.

7.2. Zero outer frame in \mathcal{V}

We discuss a simulation with \mathcal{V} matrix (equation (38))

$$\mathcal{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{8} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{4} & \frac{2}{15} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (98)$$

The sparsity pattern of \mathcal{V} educes additional conserved quantities, as discussed in section 4. These conservation laws allow us to predict the asymptotic thermal equilibrium state. Specifically, figure 4 shows the projection onto the σ_z Pauli matrix: according to equation (61), the sum of types **a** and **c** remains constant in time (red curve), but not necessarily the individual types.

Figure 5 illustrates the exponential convergence to thermal equilibrium.

7.3. Unitary rotation

We transform \mathcal{V} in equation (98) by a unitary rotation

$$\mathcal{V} \rightarrow (U^a \otimes U^c) \mathcal{V} (U^b \otimes U^d)^* \quad (99)$$

with U^a , U^c and U^d equal to the identity matrix, and

$$U^b = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \quad \text{with } \varphi = \pi/5. \quad (100)$$

This results in

$$\mathcal{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} \sin(\varphi) & -\frac{5}{8} \cos(\varphi) & \frac{1}{3} \cos(\varphi) & \frac{5}{8} \sin(\varphi) \\ \frac{2}{15} \sin(\varphi) & -\frac{1}{4} \cos(\varphi) & \frac{2}{15} \cos(\varphi) & \frac{1}{4} \sin(\varphi) \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (101)$$

with $\varphi = \pi/5$. The set of conservation laws remains unchanged ('zero outer frame in \mathcal{V} matrix', last row in table 1) when represented in the basis $(U^\alpha)^* W^\alpha(p, t) U^\alpha$, although the zero pattern is not evident from equation (101). Asymptotically, $(U^\alpha)^* W^\alpha(p, t) U^\alpha$ becomes diagonal for $t \rightarrow \infty$, which implies in this case that $W^b(p, t)$ will have non-vanishing off-diagonal entries for $t \rightarrow \infty$, as visualized in figure 6.

7.4. Effect of the conservative collision operator

Typically, the conservative collision operator C_{cons} influences the time evolution only slightly. To illustrate this observation quantitatively, we compare a simulation with the physically correct $C = C_{\text{diss}} + C_{\text{cons}}$ and a simulation using C_{diss} only. Figure 7 shows the corresponding L^1 distance between the Wigner states in dependence of time, for the interaction matrices in equations (96) and (97). One observes oscillations during the time interval [0.04, 0.1]. Note that the distance has to approach zero since the asymptotic ($t \rightarrow \infty$) thermal equilibrium state

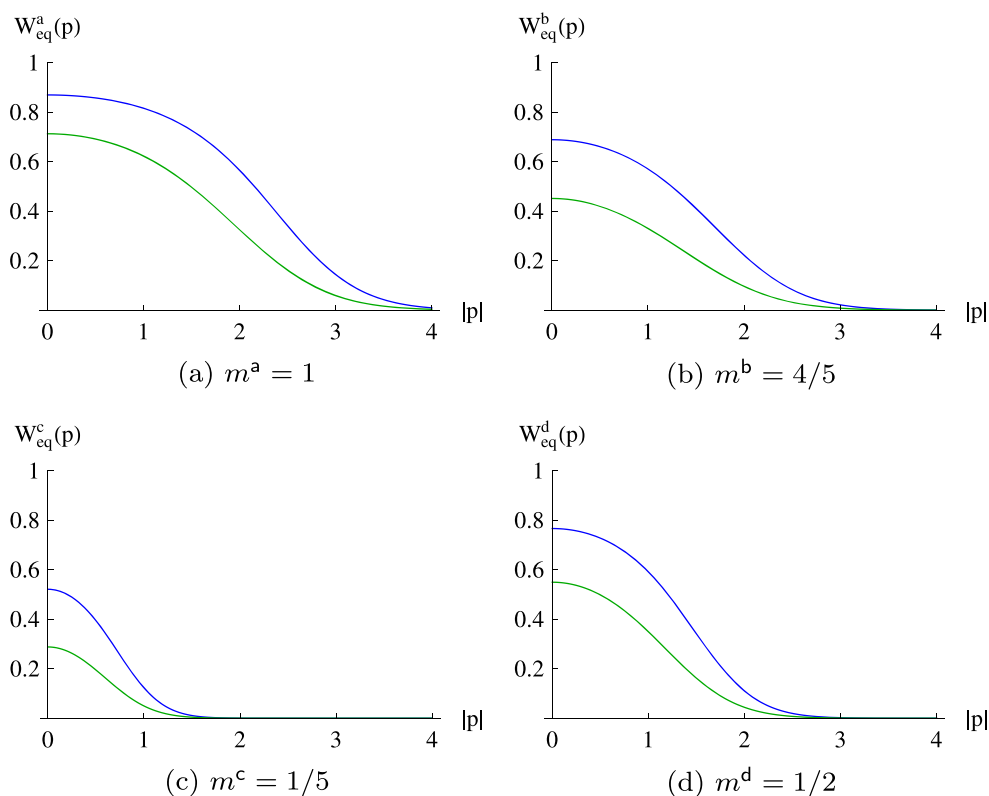


Figure 2. Diagonal matrix entries of the $t \rightarrow \infty$ thermal equilibrium Fermi–Dirac states corresponding to the initial state in figure 1, for the case of all V^{ab} matrices proportional to the identity matrix. The off-diagonal entries are zero since the states are represented in the eigenbasis of $\rho(t)$, which is conserved in this case. The common inverse temperature $\beta = 0.8193$ and the chemical potentials for each particle type have been determined from the conservation laws.

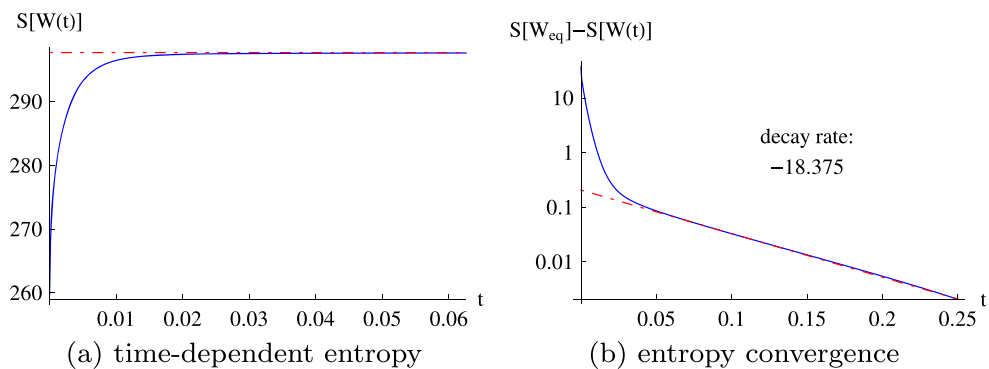


Figure 3. Entropy as function of time for the initial state in figure 1 and the β decay interaction matrices in equations (96) and (97).

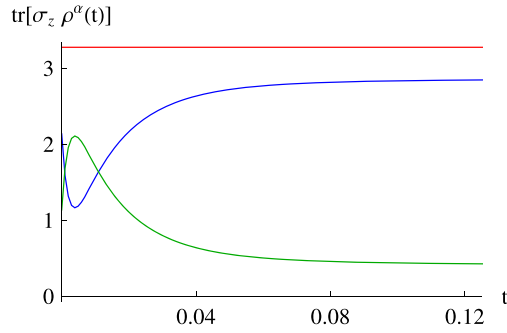


Figure 4. Projection of the density $\rho^\alpha(t)$ onto the σ_z Pauli matrix, $\text{tr}[\sigma_z \rho^\alpha(t)]$. According to equation (61), the sum over particle types **a** and **c** should be conserved (red curve), while the individual types are not necessarily constant in time (blue and green curves for **a** and **c**, respectively). The curves have been shifted for visual clarity.

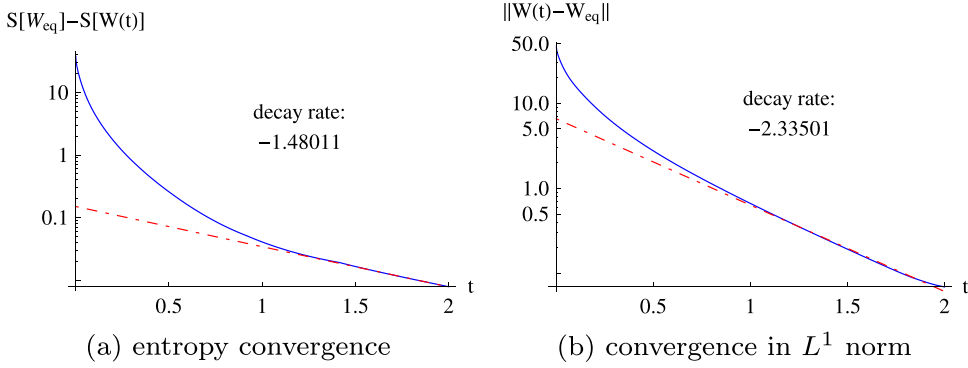


Figure 5. Exponential convergence to thermal equilibrium starting from the initial state in figure 1 and \mathcal{V} matrix in equation (98).

remains the same when omitting C_{cons} . In general terms, the trajectories of $W(t)$ are different, but share the same starting point and asymptotic thermal state.

8. Conclusions and outlook

We have disentangled the delicate relationship between the interaction matrices and the time evolution dynamics. As first insight, the interaction matrices $V^{\alpha\beta}$ enter the Boltzmann equation only via the \mathcal{V} matrix defined in equation (38). Additional conservation laws (table 1) emerge depending on the structure of \mathcal{V} . This structure is to be understood modulo unitary rotations of the form (46). The conserved quantities in turn determine the asymptotic thermal equilibrium state. Thus, while the particular matrix entries of \mathcal{V} influence the time evolution under the Boltzmann equation, only the structure class of \mathcal{V} dictates the asymptotic state. A complete characterization of all structure classes and corresponding conservation laws is still open, as well as a geometric picture of the manifold of structure classes.

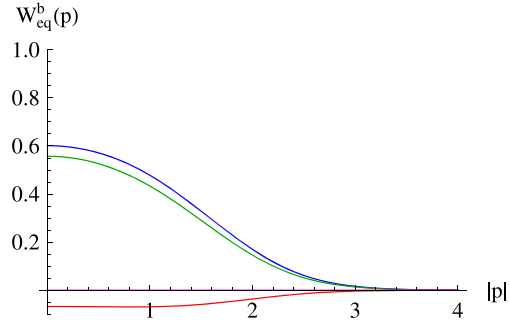


Figure 6. The asymptotic $t \rightarrow \infty$ thermal equilibrium state $W_{\text{eq}}^b(p)$ for the \mathcal{V} matrix in equation (101). The blue and green (upper) curves show the real diagonal entries, and the red curve the real part of the off-diagonal $|\uparrow\rangle\langle\downarrow|$ entry, respectively. The imaginary part of $|\uparrow\rangle\langle\downarrow|$ (magenta curve) is zero in this case. The state has non-vanishing off-diagonal entries due to the unitary rotation of the b-component in \mathcal{V} . The remaining components $W_{\text{eq}}^\alpha(p)$ for $\alpha \in \{\text{a, c, d}\}$ are diagonal.

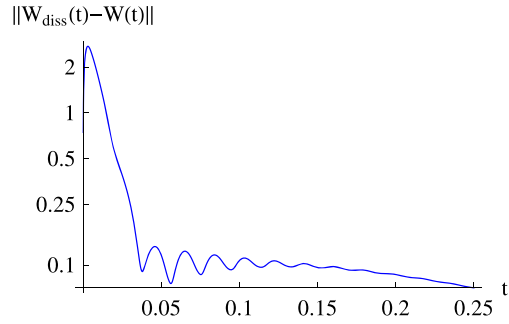


Figure 7. Distance between $W(t)$ and $W_{\text{diss}}(t)$ obtained from a simulation with C_{diss} only, for the β decay interaction matrices in equations (96) and (97).

Appendix A. Derivation of the multi-component Boltzmann equation

In this section, we derive the Boltzmann equation starting from the Hamiltonian in equation (3). In the spatially homogeneous case, the central quantity is the time-dependent two-point function

$$\langle \hat{a}_\sigma^\alpha(p, t) \hat{a}_\tau^\beta(p', t) \rangle = |U| \delta(p - p') W_{\sigma\tau}^{\alpha\beta}(p, t), \quad (102)$$

which for times up to order λ^{-2} will approximately satisfy a kinetic equation. $\langle \cdot \rangle$ denotes the average over the initial state and operators are taken to be in the Heisenberg picture $A(t) = e^{iHt} A e^{-iHt}$.

A.1. Basic definitions

Analogous to [17], we introduce spin- and field-dependent vector-valued operators

$$B_{\bar{f}}(p, t) = \sum_{\substack{\alpha \in \{a,b,c,d\}, \\ \sigma \in \{\uparrow, \downarrow\}}} B_{\sigma}^{\alpha}(p, t) \bar{f}_{\sigma}^{\alpha} \hat{e}_{\sigma}^{\alpha}, \quad (103)$$

and

$$B_{\mathbf{g}}(p, t) = \sum_{\substack{\alpha \in \{a,b,c,d\}, \\ \sigma \in \{\uparrow, \downarrow\}}} B_{\sigma}^{\alpha}(p, t) \mathbf{g}_{\sigma}^{\alpha} \hat{e}_{\sigma}^{\alpha}, \quad (104)$$

where $\bar{f}_{\sigma}^{\alpha}$ is the hermitian conjugate of the complex number f_{σ}^{α} and $\hat{e}_{\sigma}^{\alpha}$ is a unit vector. $B_{\sigma}^{\alpha}(p, t)$ is a function in momentum and time. Moreover, we introduce the inner product for spin vectors in two different spin spaces \bar{f} and \mathbf{g} as

$$B_{\bar{f}} \odot D_{\mathbf{g}} = \sum_{\alpha, \beta, \sigma, \tau} \bar{f}_{\sigma}^{\alpha} B_{\sigma}^{\alpha} D_{\tau}^{\beta} \mathbf{g}_{\tau}^{\beta}. \quad (105)$$

Thus we will always get a kind of matrix-like term. A matrix A acts on spin vectors by

$$A \cdot D_{\bar{f}} = \sum_{\alpha, \beta} \sum_{\sigma, \tau} A_{\tau\sigma}^{\alpha\beta} D_{\sigma}^{\beta} \bar{f}_{\sigma}^{\beta} = (A \cdot D)_{\bar{f}}. \quad (106)$$

Furthermore, we define the term $B_{\bar{f}}^{\alpha} \odot C_{\mathbf{g}}^{\beta}$ for particle dependent vectors as sum over spins σ and τ ,

$$B_{\bar{f}}^{\alpha} \odot D_{\mathbf{g}}^{\beta} = \sum_{\sigma, \tau \in \{\uparrow, \downarrow\}} \bar{f}_{\sigma}^{\alpha} B_{\sigma}^{\alpha} D_{\tau}^{\beta} \mathbf{g}_{\tau}^{\beta} \quad (107)$$

and spin interaction matrices

$$V_{\bar{f}}^{\alpha\beta} = \sum_{\sigma, \tau \in \{\uparrow, \downarrow\}} V_{\sigma\tau}^{\alpha\beta} \bar{f}_{\tau}^{\beta} \quad (108)$$

and

$$V_{\mathbf{g}}^{\alpha\beta} = \sum_{\sigma, \tau \in \{\uparrow, \downarrow\}} \mathbf{g}_{\sigma}^{\alpha} V_{\sigma\tau}^{\alpha\beta}. \quad (109)$$

A.2. Time evolution of the two-point correlation function

Using the introduced notation, we calculate the time evolution of

$$\frac{d}{dt} \langle \hat{a}_{\bar{f}}(p_1, t)^* \odot \hat{a}_{\mathbf{g}}(p_5, t) \rangle = \langle \dot{\hat{a}}_{\bar{f}}(p_1, t)^* \odot \hat{a}_{\mathbf{g}}(p_5, t) + \hat{a}_{\bar{f}}(p_1, t)^* \odot \dot{\hat{a}}_{\mathbf{g}}(p_5, t) \rangle \quad (110)$$

with

$$\hat{a}_{\bar{f}}(p, t)^* = \sum_{\alpha, \tau} \hat{a}_{\tau}^{\alpha}(p, t)^* \bar{f}_{\tau}^{\alpha} \hat{e}_{\tau}^{\alpha}$$

and $a_{\mathbf{g}}(p_5, t)$ respectively. The dot above a quantity $a(t)$ denotes time derivative: $\frac{d}{dt}a(t) = \dot{a}(t)$. The time derivative of a field for a single particle type is given by the Heisenberg equation of motion

$$\frac{d}{dt} \hat{a}_\tau^\alpha(p, t)^* = i[H, \hat{a}_\tau^\alpha(p, t)^*] = i e^{-iHt} [H_0 + \lambda H_1, \hat{a}_\tau^\alpha(p)^*] e^{iHt}.$$

The calculation of the H_0 part results in

$$[H_0, \hat{a}_{\bar{f}}(p)^*] = \hat{a}_{\bar{f}}(p)^* \cdot \omega(p). \quad (111)$$

Concerning the H_1 part, note that the fields depend on different momenta p_1 to p_4 . Specifically for particle type **a** one obtains

$$\begin{aligned} [H_1, \hat{a}_\tau^a(p)^*] &= -2\hat{a}_\tau^a(p)^* H_1 + \frac{1}{|U|^3} \sum_{p_{1234}} \sum_{\sigma} \delta(\underline{p}) \\ & \left[(\hat{a}_{\sigma_1}^b(p_1)^* V_{\sigma_1 \sigma_2}^{ba} \delta_{p, p_2} \delta_{\tau, \sigma_2}) (\hat{a}_{\sigma_3}^d(p_3)^* V_{\sigma_3 \sigma_4}^{dc} \hat{a}_{\sigma_4}^c(p_4)) \right. \\ & \left. + (\hat{a}_{\sigma_1}^d(p_1)^* V_{\sigma_1 \sigma_2}^{da} \delta_{p, p_2} \delta_{\tau, \sigma_2}) (\hat{a}_{\sigma_3}^b(p_3)^* V_{\sigma_3 \sigma_4}^{bc} \hat{a}_{\sigma_4}^c(p_4)) \right], \end{aligned} \quad (112)$$

where $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. For the following, we define the set

$$\begin{aligned} T = \{ & (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}), (\mathbf{a}, \mathbf{d}, \mathbf{c}, \mathbf{b}), (\mathbf{b}, \mathbf{a}, \mathbf{d}, \mathbf{c}), (\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{a}), \\ & (\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{b}, \mathbf{a}, \mathbf{d}), (\mathbf{d}, \mathbf{c}, \mathbf{b}, \mathbf{a}), (\mathbf{d}, \mathbf{a}, \mathbf{b}, \mathbf{c}) \} \end{aligned} \quad (113)$$

and

$$\begin{aligned} \mathcal{A}_{\bar{f}}[h, \mathbf{a}^*, \mathbf{a}, \mathbf{a}^*](p_1, t) &= \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}) \sum_{\alpha \in T} h^\alpha(t) \\ & \times [\mathbf{a}^{\alpha_4}(p_4, t)^* \cdot V^{\alpha_4 \alpha_3} \cdot \mathbf{a}^{\alpha_3}(p_3, t)] \mathbf{a}^{\alpha_2}(p_2, t)^* \cdot V_{\bar{f}}^{\alpha_2 \alpha_1} \end{aligned} \quad (114)$$

as well as

$$\begin{aligned} \mathcal{A}_{\bar{g}}[h, \mathbf{a}, \mathbf{a}^*, \mathbf{a}](p_1, t) &= \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}) \sum_{\alpha \in T} h^\alpha(t) V_{\bar{g}}^{\alpha_1 \alpha_2} \cdot \mathbf{a}^{\alpha_2}(p_2, t) \\ & \times [\mathbf{a}^{\alpha_3}(p_3, t)^* \cdot V^{\alpha_3 \alpha_4} \cdot \mathbf{a}^{\alpha_4}(p_4, t)]. \end{aligned} \quad (115)$$

Using the invariance under interchanges $p_1 \leftrightarrow p_3$, $p_2 \leftrightarrow p_4$ as well as $(p_1, p_3) \leftrightarrow (p_4, p_2)$, we are able to write the time derivatives of the creation and annihilation operators as

$$\frac{d}{dt} \hat{a}_{\bar{f}}(p_1, t)^* = i \hat{a}_{\bar{f}}(p_1, t)^* \cdot \omega(p_1) - 2i\lambda \hat{a}_{\bar{f}}(p_1, t)^* H_1(t) + i\lambda \mathcal{A}_{\bar{f}}[\text{id}, \hat{a}^*, \hat{a}, \hat{a}^*](p_1, t) \quad (116)$$

and

$$\frac{d}{dt} \hat{a}_{\bar{g}}(p_1, t) = -i\omega(p_1) \cdot \hat{a}_{\bar{g}}(p_1, t) + 2i\lambda H_1(t) \hat{a}_{\bar{g}}(p_1, t) - i\lambda \mathcal{A}_{\bar{g}}[\text{id}, \hat{a}, \hat{a}^*, \hat{a}](p_1, t), \quad (117)$$

where id denotes the identity function. In order to simplify calculations, we switch to the interaction picture and define

$$\mathbf{a}_{\bar{f}}(p, t)^* = \hat{a}_{\bar{f}}(p, t)^* \cdot e^{-i\omega(p)t} e^{i2\lambda \int_0^t ds H_1(s)}$$

and

$$\mathbf{a}_{\bar{g}}(p, t) = e^{-i2\lambda \int_0^t ds H_1(s)} e^{i\omega(p)t} \cdot \hat{a}_{\bar{g}}(p, t),$$

respectively. Thus the dynamics of $\mathbf{a}(p, t)$ is given by

$$\frac{d}{dt} \mathbf{a}_{\bar{f}}(p_1, t)^* = i\lambda \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \mathbf{a}^*, \mathbf{a}, \mathbf{a}^*](p_1, t) \quad (118)$$

and

$$\frac{d}{dt} \mathbf{a}_{\underline{g}}(p, t) = -i\lambda \mathcal{A}_{\underline{g}}[e^{i\omega_{1234}t}, \mathbf{a}, \mathbf{a}^*, \mathbf{a}](p_1, t). \quad (119)$$

Moreover

$$\begin{aligned} \mathcal{A}[e^{-i\omega_{1234}t}, \mathbf{a}^*, \mathbf{a}, \mathbf{a}^*](p_1, t) &= \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}) \sum_{\alpha \in T} e^{-i\omega_{1234}t} \\ &\times [\mathbf{a}^{\alpha_4}(p_4, t)^* \cdot V^{\alpha_4 \alpha_3} \cdot \mathbf{a}^{\alpha_3}(p_3, t)] \mathbf{a}^{\alpha_2}(p_2, t)^* \cdot V^{\alpha_2 \alpha_1} \end{aligned} \quad (120)$$

and

$$\begin{aligned} \mathcal{A}[e^{i\omega_{1234}t}, \mathbf{a}, \mathbf{a}^*, \mathbf{a}](p_1, t) &= \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}) \sum_{\alpha \in T} V^{\alpha_1 \alpha_2} \cdot \mathbf{a}^{\alpha_2}(p_2, t) \\ &\times [\mathbf{a}^{\alpha_3}(p_3, t)^* \cdot V^{\alpha_3 \alpha_4} \cdot \mathbf{a}^{\alpha_4}(p_4, t)]. \end{aligned} \quad (121)$$

A.3. Expansion in powers of λ

Iteration of (118) and (119) twice up to second order leads to

$$\mathbf{a}_{\bar{f}}(p_1, t)^* = \mathbf{a}_{\bar{f}}(p_1, 0)^* + i\lambda \int_0^t ds \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}s}, \mathbf{a}^*, \mathbf{a}, \mathbf{a}^*](p_1, s) \quad (122)$$

and carrying out the iteration up to order λ^2 (Duhamel expansion)

$$\begin{aligned} \frac{d}{dt} \mathbf{a}_{\bar{f}}(p_1, t)^* &= i\lambda \mathcal{A}[e^{-i\omega_{1234}t}, \hat{\mathbf{a}}^*, \hat{\mathbf{a}}, \hat{\mathbf{a}}^*](p_1, 0) \\ &\quad - \lambda^2 \int_0^t ds \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \mathcal{A}[e^{-i\omega_{4678}s}, \hat{\mathbf{a}}^*, \hat{\mathbf{a}}, \hat{\mathbf{a}}^*], \hat{\mathbf{a}}, \hat{\mathbf{a}}^*](p_1, s) \\ &\quad + \lambda^2 \int_0^t ds \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \hat{\mathbf{a}}^*, \mathcal{A}[e^{i\omega_{3678}s}, \hat{\mathbf{a}}, \hat{\mathbf{a}}^*, \hat{\mathbf{a}}], \hat{\mathbf{a}}^*](p_1, s) \\ &\quad - \lambda^2 \int_0^t ds \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \hat{\mathbf{a}}^*, \hat{\mathbf{a}}, \mathcal{A}[e^{-i\omega_{2678}s}, \hat{\mathbf{a}}^*, \hat{\mathbf{a}}, \hat{\mathbf{a}}^*]](p_1, s) \\ &\quad + \mathcal{O}(\lambda^3) \\ &= \lambda \mathbf{a}_{\bar{f}}^{(1)}(p, t)^* + \lambda^2 \mathbf{a}_{\bar{f}}^{(2)}(p, t)^* + \mathcal{O}(\lambda^3), \end{aligned} \quad (123)$$

where $\mathbf{a}_{\bar{f}}^{(i)}(p, t)$ refers to the terms of order λ^i .

Note that the first term, $\mathbf{a}_{\bar{f}}^{(0)}(p, t)^*$, reflects the zero point of the integration and therefore reads

$$\mathbf{a}_{\bar{f}}^{(0)}(p, 0)^* = \hat{\mathbf{a}}_{\bar{f}}(p)^*. \quad (124)$$

Furthermore, the following identity holds

$$\langle \hat{a}_{\bar{f}}(p_1, t)^* \odot a_g(p_5, t) \rangle = \langle \hat{a}_{\bar{f}}(p_1, t)^* \odot \hat{a}_g(p_5, t) \rangle. \quad (125)$$

Iterating further gives

$$\begin{aligned} \langle \hat{a}_{\bar{f}}(p_1, t)^* \odot \hat{a}_g(p_5, t) \rangle &= \langle a_{\bar{f}}(p_1, 0) \odot a_g(p_5, 0) \rangle \\ &+ \sum_{n=1}^{\infty} \lambda^n \sum_{m=0}^n \langle a_{\bar{f}}^{(m)}(p_1, t)^* \odot a_g^{(n-m)}(p_5, t) \rangle \\ &= \delta(p_1 - p_5) \sum_{n=0}^{\infty} \lambda^n \langle \bar{f}, W^{(n)}(p_1, t) \cdot g \rangle, \end{aligned} \quad (126)$$

where $W^{(n)}(p_1, t)$ is a summation of the relevant terms for λ^n .

A.3.1. First-order terms. Starting with the linear λ terms, the first thing to do is to calculate $a_{\bar{f}}^{(1)}(p_1, t)$ exactly. Therefore, $a^\alpha(p, t)$ and $a^\alpha(p, t)^*$ in (115) and (114) have to be replaced by $\hat{a}^\alpha(p)$ and $\hat{a}^\alpha(p)^*$. The result is

$$\begin{aligned} &\delta(p_1 - p_5) \langle \bar{f}, W^{(1)}(p_1, t) \cdot g \rangle \\ &= \langle a_{\bar{f}}^0(p_1, t)^* \odot a_g^{(0)}(p_5, 0) + a_{\bar{f}}^{(0)}(p_1, 0)^* \odot a_g^{(1)}(p_5, t) \rangle \\ &= i \langle \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \hat{a}^*, \hat{a}, \hat{a}^*](p_1, 0)^* \odot \hat{a}_g(p_5) \\ &\quad - a_{\bar{f}}(p_1)^* \odot \mathcal{A}_g[e^{i\omega_{1234}t}, \hat{a}, \hat{a}^*, \hat{a}](p_5, 0) \rangle. \end{aligned} \quad (127)$$

Using equation (114) on the first term we get

$$\begin{aligned} &\langle \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \hat{a}^*, \hat{a}, \hat{a}^*](p_1, 0)^* \odot \hat{a}_g(p_5) \rangle \\ &= i \int_0^t ds \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}) \sum_{\alpha \in T} \sum_{\beta} e^{-i\omega_{234}^{\alpha} s} \langle [\hat{a}^{\alpha_4}(p_4)^* \cdot V^{\alpha_4 \alpha_3} \cdot \hat{a}^{\alpha_3}(p_3)] \\ &\quad \times [(\hat{a}^{\alpha_2}(p_2)^* \cdot V_{\bar{f}}^{\alpha_2 \alpha_1}) \odot \hat{a}_g^{\beta}(p_5)] \rangle. \end{aligned} \quad (128)$$

Each summand in $\alpha \in T$ and β can be represented by a graph, see [17].

Now, to form the average value of equation (127) via equation (102), we have to perform Wick contractions. If we are averaging over an initial quasi-free state we can partition this average into a product of averages containing only two operators by using the following rule

$$\langle \hat{a}_{i_1}^* \hat{a}_{j_1} \cdots \hat{a}_{i_n}^* \hat{a}_{j_n} \rangle = \det [K(i_k, j_l)]_{1 \leq k, l \leq n}, \quad (129)$$

where

$$K(i_k, j_l) = \begin{cases} \langle \hat{a}_{i_k}^* \hat{a}_{j_l} \rangle, & \text{if } k \leq l, \\ -\langle \hat{a}_{j_l} \hat{a}_{i_k}^* \rangle, & \text{if } k > l. \end{cases} \quad (130)$$

One obtains, for example

$$\begin{aligned}
& \left\langle \hat{a}_{\sigma_4}^d(p_4)^* \hat{a}_{\sigma_3}^c(p_3) \hat{a}_{\sigma_2}^b(p_2)^* \hat{a}_\tau^\beta(p_5) \right\rangle \\
&= \det \begin{bmatrix} \left\langle \hat{a}_{\sigma_4}^d(p_4)^* \hat{a}_{\sigma_3}^c(p_3) \right\rangle & \left\langle \hat{a}_{\sigma_4}^d(p_4)^* \hat{a}_\tau^\beta(p_5) \right\rangle \\ -\left\langle \hat{a}_{\sigma_3}^c(p_3) \hat{a}_{\sigma_2}^b(p_2)^* \right\rangle & \left\langle \hat{a}_{\sigma_2}^b(p_2)^* \hat{a}_\tau^\beta(p_5) \right\rangle \end{bmatrix} \\
&= \left\langle \hat{a}_{\sigma_4}^d(p_4)^* \hat{a}_{\sigma_3}^c(p_3) \right\rangle \left\langle \hat{a}_{\sigma_2}^b(p_2)^* \hat{a}_\tau^\beta(p_5) \right\rangle \\
&\quad + \left\langle \hat{a}_{\sigma_4}^d(p_4)^* \hat{a}_\tau^\beta(p_5) \right\rangle \left\langle \hat{a}_{\sigma_3}^c(p_3) \hat{a}_{\sigma_2}^b(p_2)^* \right\rangle \\
&= 0
\end{aligned} \tag{131}$$

for all $\beta \in \{a, b, c, d\}$ since the average value over a pair of annihilator and creator of different particle types is

$$\left\langle \hat{a}_\sigma^\alpha(p)^* \hat{a}_\tau^\beta(p') \right\rangle = \left\langle \hat{a}_\tau^\beta(p') \hat{a}_\sigma^\alpha(p)^* \right\rangle = 0. \tag{132}$$

Similarly all λ terms of order one are zero, and therefore equation (126) reduces to

$$\begin{aligned}
& \int_0^t ds \frac{d}{ds} \left\langle \hat{a}_{\bar{f}}(p_1, s)^* \odot \hat{a}_{\bar{g}}(p_5, s) \right\rangle \\
&= \lambda^2 \delta(p_1 - p_5) \int_0^t ds \left\langle \bar{f}, W^{(2)}(p_1, s) \cdot \bar{g} \right\rangle + \mathcal{O}(\lambda^3).
\end{aligned} \tag{133}$$

A.3.2. Second-order terms. The full $\delta(p_1 - p_5) \lambda^2 \langle \bar{f}, W^{(2)}(p_1, t) \cdot \bar{g} \rangle$ reads

$$\begin{aligned}
& \delta(p_1 - p_5) \lambda^2 \left\langle \bar{f}, W^{(2)}(p_1, t) \cdot \bar{g} \right\rangle \\
&= \int_0^t ds \left\langle \alpha_{\bar{f}}^{(1)}(p_1, t)^* \odot \alpha_{\bar{g}}^{(1)}(p_5, s) + \alpha_{\bar{f}}^{(1)}(p_1, s)^* \odot \alpha_{\bar{g}}^{(1)}(p_5, t) \right\rangle \\
&\quad + \left\langle \alpha_{\bar{f}}^{(2)}(p_1, t)^* \odot \hat{a}_{\bar{g}}(p_5) + \hat{a}_{\bar{f}}(p_1)^* \odot \alpha_{\bar{g}}^{(2)}(p_5, t) \right\rangle.
\end{aligned} \tag{134}$$

Explicitly, the first (1)(1) term is given by

$$\alpha_{\bar{f}}^{(1)}(p_1, t)^* \odot \alpha_{\bar{g}}^{(1)}(p_5, s) = \mathcal{A}_{\bar{f}} \left[e^{-i\omega_{1234}t}, \hat{a}^*, \hat{a}, \hat{a}^* \right] (p_1, t) \odot \mathcal{A}_{\bar{g}} \left[e^{i\omega_{1234}s}, \hat{a}, \hat{a}^*, \hat{a} \right] (p_5, s) \tag{135}$$

and the second (1)(1) term results from interchanging $s \leftrightarrow t$. We get

$$\begin{aligned}
\alpha_{\bar{f}}^{(1)}(p_1, t)^* \odot \alpha_{\bar{g}}^{(1)}(p_5, s) &= \frac{1}{|U|^6} \sum_{p_{234}, p_{678}} \delta(p_{1234}) \delta(p_{5678}) \sum_{\alpha, \beta \in T} e^{-i\omega_{1234}t} e^{i\omega_{5678}s} \\
&\quad \left(\hat{a}^{\alpha_4}(p_4)^* \cdot V^{\alpha_4 \alpha_3} \cdot \hat{a}^{\alpha_3}(p_3) \right) \left(\hat{a}^{\alpha_2}(p_2)^* \cdot V_{\bar{f}}^{\alpha_2 \alpha_1} \right) \\
&\quad \odot \left(V_{\bar{g}}^{\beta_1 \beta_2} \cdot \hat{a}^{\beta_2}(p_6) \right) \left(\hat{a}^{\beta_3}(p_7)^* \cdot V^{\beta_3 \beta_4} \cdot \hat{a}^{\beta_4}(p_8) \right).
\end{aligned} \tag{136}$$

For what follows, we assume that the initial state $\langle \cdot \rangle$ is quasifree, gauge invariant and invariant under translations. Then the two-point function $\langle a_\sigma^\alpha(p)^* a_\tau^\beta(p') \rangle$ is determined by

$$\left\langle a_{\sigma}^{\alpha}(p)^* a_{\tau}^{\beta}(p') \right\rangle = |U| \delta_{\alpha\beta} \delta(p - p') W_{\sigma\tau}^{\alpha}(p). \quad (137)$$

After taking the average $\langle \cdot \rangle$, the summand on the right of equation (136) with $\beta_1 = \alpha_1 = \mathbf{a}$ is given by

$$\begin{aligned} & \frac{1}{|U|^3} \sum_{p_{234}} \delta(p_{1234}) \delta(p_{5234}) e^{-i\omega_{1234}^{abcd} t} e^{i\omega_{5234}^{abcd} s} \\ & \left[\left\langle f^{\mathbf{a}}, V^{\mathbf{ab}} \cdot W_2^{\mathbf{b}} \cdot V^{\mathbf{ba}} \cdot g^{\mathbf{a}} \right\rangle \text{tr} \left[V^{\mathbf{cd}} \cdot W_4^{\mathbf{d}} \cdot V^{\mathbf{dc}} \cdot \tilde{W}_3^{\mathbf{c}} \right] \right. \\ & + \left\langle f^{\mathbf{a}}, V^{\mathbf{ad}} \cdot W_4^{\mathbf{d}} \cdot V^{\mathbf{da}} \cdot g^{\mathbf{a}} \right\rangle \text{tr} \left[V^{\mathbf{cb}} \cdot W_2^{\mathbf{b}} \cdot V^{\mathbf{bc}} \cdot \tilde{W}_3^{\mathbf{c}} \right] \\ & + \left\langle f^{\mathbf{a}}, V^{\mathbf{ab}} \cdot W_2^{\mathbf{b}} \cdot V^{\mathbf{bc}} \cdot \tilde{W}_3^{\mathbf{c}} \cdot V^{\mathbf{cd}} \cdot W_4^{\mathbf{d}} \cdot V^{\mathbf{da}} \cdot g^{\mathbf{a}} \right\rangle \\ & \left. + \left\langle f^{\mathbf{a}}, V^{\mathbf{ad}} \cdot W_4^{\mathbf{d}} \cdot V^{\mathbf{dc}} \cdot \tilde{W}_3^{\mathbf{c}} \cdot V^{\mathbf{cb}} \cdot W_2^{\mathbf{b}} \cdot V^{\mathbf{ba}} \cdot g^{\mathbf{a}} \right\rangle \right], \quad (138) \end{aligned}$$

where $f^{\mathbf{a}}$ and $g^{\mathbf{a}}$ are defined as

$$f^{\mathbf{a}} = (f_{\uparrow}^{\mathbf{a}}, f_{\downarrow}^{\mathbf{a}})^T, \quad g^{\mathbf{a}} = (g_{\uparrow}^{\mathbf{a}}, g_{\downarrow}^{\mathbf{a}})^T. \quad (139)$$

The (b, b), (c, c), and (d, d) components are analogous. We obtain the (b, b) component by interchanging $\mathbf{a} \leftrightarrow \mathbf{b}$, $\mathbf{c} \leftrightarrow \mathbf{d}$, the (c, c) component by interchanging $\mathbf{a} \leftrightarrow \mathbf{c}$, $\mathbf{b} \leftrightarrow \mathbf{d}$ and the (d, d) component by interchanging $\mathbf{a} \leftrightarrow \mathbf{d}$, $\mathbf{b} \leftrightarrow \mathbf{c}$.

We collect the components of the Wigner states in a 8×8 block-diagonal matrix

$$W_1 = \text{diag} \left[W_1^{\mathbf{a}}, W_1^{\mathbf{b}}, W_1^{\mathbf{c}}, W_1^{\mathbf{d}} \right], \quad (140)$$

where each entry stands for a 2×2 -matrix. The interaction potential is summarized by the matrices $V^=$ and V^{\times} defined in equations (36) and (37), respectively. For the following, define

$$G_{ij}^{\alpha\beta} = \text{tr} \left[V^{\beta\alpha} \cdot W_j^{\alpha} \cdot V^{\alpha\beta} \cdot \tilde{W}_i^{\beta} \right] = \left(\mathcal{G}_{ij}^{\alpha\beta} \right)^*. \quad (141)$$

This definition is used in

$$\mathcal{G}_{ij}^{\alpha\beta\gamma\delta} = \text{diag} \left[G_{ij}^{\delta\gamma}, G_{ij}^{\gamma\delta}, G_{ij}^{\beta\alpha}, G_{ij}^{\alpha\beta} \right] \otimes \mathbb{1}_{2 \times 2} \quad (142)$$

and

$$\hat{\mathcal{G}}_{ij}^{\alpha\beta\gamma\delta} = \text{diag} \left[G_{ij}^{\delta\gamma}, G_{ij}^{\alpha\beta}, G_{ij}^{\beta\alpha}, G_{ij}^{\gamma\delta} \right] \otimes \mathbb{1}_{2 \times 2}, \quad (143)$$

where the components two and four are exchanged. With these definitions

$$\begin{aligned} \left\langle a_{\uparrow}^{(1)}(p_1, t)^* \odot a_{\uparrow}^{(1)}(p_5, s) \right\rangle &= \frac{1}{|U|^3} \sum_{p_{234}} \delta(p_{1234}) \delta(p_{5234}) \\ & \times \left\langle f, e^{-i\omega_{1234} t} \cdot e^{i\omega_{5234} s} \cdot \left(\mathcal{G}_{34}^{\mathbf{abcd}} \cdot V^= \cdot W_2 \cdot V^= + \hat{\mathcal{G}}_{32}^{\mathbf{adcb}} \cdot V^{\times} \cdot W_4 \cdot V^{\times} \right. \right. \\ & + V^{\times} \cdot W_4 \cdot V^= \cdot \tilde{W}_3 \cdot V^{\times} \cdot W_2 \cdot V^= \\ & \left. + V^= \cdot W_2 \cdot V^{\times} \cdot \tilde{W}_3 \cdot V^= \cdot W_4 \cdot V^{\times} \right) \cdot g \rangle. \quad (144) \end{aligned}$$

Furthermore, we define

$$\begin{aligned} \mathcal{D}[W]_{234} &= \mathcal{G}_{34}^{\text{abcd}} \cdot V^= \cdot W_2 \cdot V^= + \hat{\mathcal{G}}_{32}^{\text{adcb}} \cdot V^\times \cdot W_4 \cdot V^\times \\ &\quad + V^\times \cdot W_4 \cdot V^= \cdot \tilde{W}_3 \cdot V^\times \cdot W_2 \cdot V^= \\ &\quad + V^= \cdot W_2 \cdot V^\times \cdot \tilde{W}_3 \cdot V^= \cdot W_4 \cdot V^\times. \end{aligned} \quad (145)$$

By an analogous calculation

$$\begin{aligned} \left\langle \mathbf{a}_{\bar{f}}^{(1)}(p_1, s)^* \cdot \mathbf{a}_{\bar{g}}^{(1)}(p_5, t) \right\rangle &= \frac{1}{|U|^3} \sum_{p_2, p_3, p_4} \delta(\underline{p}_{1234}) \delta(\underline{p}_{5234}) \\ &\quad \times \left\langle \mathbf{f}, e^{-i\omega_{1234}s} \cdot e^{i\omega_{5234}t} \cdot \mathcal{D}[W]_{234} \cdot \mathbf{g} \right\rangle. \end{aligned} \quad (146)$$

To further simplify the expression, we rearrange the delta functions $\delta(\underline{p}_{1234})\delta(\underline{p}_{5234}) = \delta(p_1 - p_5)\delta(\underline{p}_{1234})$ and p_5 can be replaced by p_1 , such that the exponents of the exponential function change signs.

The (2)(0) term is given by

$$\begin{aligned} \mathbf{a}_{\bar{f}}^{(2)}(p_1, t)^* \odot \hat{a}_{\bar{g}}(p_5) &= -\int_0^t ds \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \mathcal{A}[e^{-i\omega_{4678}s}, \hat{a}^*, \hat{a}, \hat{a}^*]^*, \hat{a}, \hat{a}^*](p_1, s)^* \\ &\quad - \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \hat{a}^*, \mathcal{A}[e^{i\omega_{3678}s}, \hat{a}, \hat{a}^*, \hat{a}], \hat{a}^*](p_1, s)^* \\ &\quad + \mathcal{A}_{\bar{f}}[e^{-i\omega_{1234}t}, \hat{a}^*, \hat{a}, \mathcal{A}[e^{-i\omega_{2678}s}, \hat{a}^*, \hat{a}, \hat{a}^*]^*](p_1, s)^* \odot \hat{a}_{\bar{g}}(p_5). \end{aligned} \quad (147)$$

Thus we get

$$\begin{aligned} \mathbf{a}_{\bar{f}}^{(2)}(p_1, t)^* \odot \hat{a}_{\bar{g}}(p_5) &= -\frac{1}{|U|^6} \sum_{p_{234}, p_{678}} \delta(\underline{p}_{1234}) \\ &\quad \times \sum_{\beta} \sum_{\alpha \in T} \int_0^t ds e^{-i\omega_{1234}t} (X - Y + Z) \odot \hat{a}_{\bar{g}}^{\beta}(p_5) \end{aligned} \quad (148)$$

with

$$\begin{aligned} X &= \sum_{(\alpha_2, \beta_2, \beta_3, \beta_4) \in T} \delta(\underline{p}_{2678}) e^{-i\omega_{2678}^{\alpha_2 \beta_2 \beta_3 \beta_4} s} \\ &\quad \left(\hat{a}^{\alpha_4}(p_4)^* \cdot V^{\alpha_4 \alpha_3} \cdot \hat{a}^{\alpha_3}(p_3) \right) \left(\hat{a}^{\beta_4}(p_8)^* \cdot V^{\beta_4 \beta_3} \cdot \hat{a}^{\beta_3}(p_7) \right) \\ &\quad \left(\hat{a}^{\beta_2}(p_6)^* \cdot V^{\beta_2 \alpha_2} \right) V_{\bar{f}}^{\alpha_2 \alpha_1} \end{aligned} \quad (149)$$

and

$$\begin{aligned} Y &= \sum_{(\alpha_3, \beta_2, \beta_3, \beta_4) \in T} \delta(\underline{p}_{3678}) e^{i\omega_{3678}^{\alpha_3 \beta_2 \beta_3 \beta_4} s} \left(\hat{a}_{\alpha_4}^{\alpha_4}(p_4)^* \cdot V^{\alpha_4 \alpha_3} \cdot V^{\alpha_3 \beta_2} \cdot \hat{a}^{\beta_2}(p_6) \right) \\ &\quad \left(\hat{a}^{\beta_3}(p_7)^* \cdot V^{\beta_3 \beta_4} \cdot \hat{a}^{\beta_4}(p_8) \right) \hat{a}^{\alpha_2}(p_2)^* \cdot V_{\bar{f}}^{\alpha_2 \alpha_1} \end{aligned} \quad (150)$$

and

$$Z = \sum_{(\alpha_4, \beta_2, \beta_3, \beta_4) \in T} \delta(\underline{p}_{4678}) e^{-i\omega_{4678}^{\alpha_4 \beta_2 \beta_3 \beta_4} s} \left(\hat{a}^{\beta_4}(p_8)^* \cdot V^{\beta_4 \beta_3} \cdot \hat{a}^{\beta_3}(p_7) \right) \left(\hat{a}^{\beta_2}(p_6)^* \cdot V^{\beta_2 \alpha_4} \cdot V^{\alpha_4 \alpha_3} \cdot \hat{a}^{\alpha_3}(p_3) \right) \cdot \hat{a}^{\alpha_2}(p_2)^* V_{\bar{f}}^{\alpha_2 \alpha_1}. \quad (151)$$

Only terms with complementary creation and annihilation operators of the same particle type are non-zero when taking the average. For the following, we introduce

$$F_i^{\alpha\beta} = \text{tr} \left[V^{\beta\alpha} \cdot W_i^\alpha \cdot V^{\alpha\beta} \right] = \left(\mathcal{F}_i^{\alpha\beta} \right)^* \quad (152)$$

which is summarized by the block-diagonal matrices

$$\mathcal{F}_i^{\alpha\beta\gamma\delta} = \text{diag} \left[F_i^{\delta\gamma}, F_i^{\gamma\delta}, F_i^{\beta\alpha}, F_i^{\alpha\beta} \right] \otimes \mathbb{1}_{2 \times 2} \quad (153)$$

and

$$\hat{\mathcal{F}}_i^{\alpha\beta\gamma\delta} = \text{diag} \left[F_i^{\delta\gamma}, F_i^{\alpha\beta}, F_i^{\beta\alpha}, F_i^{\gamma\delta} \right] \otimes \mathbb{1}_{2 \times 2}. \quad (154)$$

Note that the second and fourth entry on the right in (154) are exchanged as compared to (153). As heuristic motivation, the calculation for the H_1^{abcd} -part in the Hamiltonian is analogous to the H_1^{adcb} -part with particles b and d exchanged. In summary, one obtains

$$\begin{aligned} \langle \mathfrak{a}_{\bar{f}}^{(2)}(p_1, t)^* \odot \hat{a}_{\mathfrak{g}}(p_5) \rangle &= \delta(p_1 - p_5) \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}_{1234}) \\ &\times \int_0^t ds \langle \mathfrak{f}, e^{-i\omega_{1234}(t-s)} \cdot \mathcal{B}[W]_{1234} \cdot \mathfrak{g} \rangle \end{aligned} \quad (155)$$

with the definition

$$\begin{aligned} \mathcal{B}[W]_{1234} &= - \left(\mathcal{G}_{34}^{\text{abcd}} \cdot V^\cdot \cdot V^\cdot \cdot W_5 + \hat{\mathcal{G}}_{32}^{\text{adcb}} \cdot V^\times \cdot V^\times \cdot W_5 \right. \\ &\quad + V^\cdot \cdot V^\times \cdot \tilde{W}_3 \cdot V^\cdot \cdot W_4 \cdot V^\times \cdot W_5 \\ &\quad + V^\times \cdot V^\cdot \cdot \tilde{W}_3 \cdot V^\times \cdot W_2 \cdot V^\cdot \cdot W_5) \\ &\quad + \left(\mathcal{F}_4^{\text{abcd}} \cdot V^\cdot \cdot \tilde{W}_2 \cdot V^\cdot \cdot W_5 + \hat{\mathcal{F}}_2^{\text{adcb}} \cdot V^\times \cdot \tilde{W}_4 \cdot V^\times \cdot W_5 \right. \\ &\quad + V^\cdot \cdot \tilde{W}_2 \cdot V^\times \cdot V^\cdot \cdot W_4 \cdot V^\times \cdot W_5 \\ &\quad + V^\times \cdot \tilde{W}_4 \cdot V^\cdot \cdot V^\times \cdot W_2 \cdot V^\cdot \cdot W_5) \\ &\quad - \left(\mathcal{F}_3^{\text{badc}} \cdot V^\cdot \cdot \tilde{W}_2 \cdot V^\cdot \cdot W_5 + \hat{\mathcal{F}}_3^{\text{dabc}} \cdot V^\times \cdot \tilde{W}_4 \cdot V^\times \cdot W_5 \right. \\ &\quad + V^\cdot \cdot \tilde{W}_2 \cdot V^\times \cdot W_3 \cdot V^\cdot \cdot V^\times \cdot W_5 \\ &\quad \left. + V^\times \cdot \tilde{W}_4 \cdot V^\cdot \cdot W_3 \cdot V^\times \cdot V^\cdot \cdot W_5 \right). \end{aligned} \quad (156)$$

For the (0)(2) term of equation (134) we get analogously

$$\langle \hat{\mathfrak{a}}_{\bar{f}}(p_1)^* \odot \mathfrak{a}_{\mathfrak{g}}^{(2)}(p_5, t) \rangle = \delta(p_1 - p_5) \frac{1}{|U|^3} \sum_{p_{234}} \delta(\underline{p}_{1234}) \int_0^s ds \langle \mathfrak{f}, e^{i\omega_{5234}(t-s)} \cdot \mathcal{B}[W]_{1234}^* \rangle. \quad (157)$$

A.4. The limit $\lambda \rightarrow 0$, $t = \mathcal{O}(\lambda^{-2})$

We take the infinite volume limit $\ell \rightarrow \infty$ of $U = U_\ell = [-\ell, \ell]^d$ and subsequently the kinetic limit $\lambda \rightarrow 0$ together with rescaling $t \rightarrow \lambda^{-2}t$. Defining

$$H(p_{1234}, t) = \int_0^t ds \int_0^s ds' \delta(\underline{p}_{1234}) \left(e^{i\omega_{1234}(s-s')} \cdot (D[W]_{234} + \mathcal{B}[W]_{1234}^*) \right. \\ \left. + e^{-i\omega_{1234}(s-s')} \cdot (D[W]_{234} + \mathcal{B}[W]_{1234}) \right), \quad (158)$$

we get

$$\delta(p_1 - p_5) \langle f, W^{(2)}(p_1, t) \cdot g \rangle \\ = \int_0^t ds \frac{d}{ds} \sum_{m=0}^2 \langle \hat{a}_f^*(p_1, s)^{(m)} \odot \hat{a}_g(p_5, s)^{(2-m)} \rangle \\ = \delta(p_1 - p_5) \frac{1}{|U_\ell|^3} \sum_{p_{234} \in \hat{U}^3} \langle f, H(p_{1234}, t) \cdot g \rangle. \quad (159)$$

In the limit $\ell \rightarrow \infty$ we obtain the Riemann integral

$$\lim_{\ell \rightarrow \infty} \frac{1}{|U_\ell|} \sum_{p \in \frac{\ell}{7} \mathbb{Z}^d} f(p) = \frac{1}{2\pi} \int_{\mathbb{R}^d} dp f(p). \quad (160)$$

Thus

$$W^{(2)}(p_1, t) = \lim_{\ell \rightarrow \infty} \frac{1}{|U_\ell|^3} \sum_{p_{234} \in \frac{\ell}{7} \mathbb{Z}^{3d}} H(p_{1234}, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^{3d}} dp_{234} H(p_{1234}, t). \quad (161)$$

The collision operator is determined by taking at second order the limit $\lambda \rightarrow 0$ and simultaneously long times $\lambda^{-2}t$ with t of order 1. More explicitly

$$t C[W](p) = \lim_{\lambda \rightarrow 0} \lambda^2 W^{(2)}(p, \lambda^{-2}t). \quad (162)$$

To evaluate the limit, we use

$$\lim_{\lambda \rightarrow 0} \lambda^2 \int_0^{\lambda^{-2}t} ds \int_0^s ds' e^{\pm i\omega_{1234}(s-s')} \\ = t \int_0^\infty ds e^{\pm i\omega_{1234}s} = t \left(\pm i \mathcal{P}(\omega_{1234}^{-1}) + \pi \delta(\omega_{1234}) \right), \quad (163)$$

where \mathcal{P} denotes the principal value integral. Thus

$$\lim_{\lambda \rightarrow 0} \lambda^2 W^{(2)}(p_1, \lambda^{-2}t) \\ = \frac{t \pi}{(2\pi)^3} \int_{\mathbb{R}^{3d}} dp_{234} \delta(\underline{p}) \delta(\omega_{1234}) \cdot (2 D[W]_{234} + \mathcal{B}[W]_{1234}^* + \mathcal{B}[W]_{1234}) \\ + \frac{t i}{(2\pi)^3} \int_{\mathbb{R}^{3d}} dp_{234} \delta(\underline{p}) \mathcal{P}(\omega_{1234}^{-1}) \cdot (\mathcal{B}[W]_{1234}^* - \mathcal{B}[W]_{1234}), \quad (164)$$

where $\mathcal{P}(\omega_{1234}^{-1})$ must be considered as principal value applied to every component and similarly $\delta(\omega_{1234})$ as a matrix of delta functions

$$\mathcal{P}(\underline{\omega}_{1234}^{-1}) = \text{diag} \left[\mathcal{P} \left(\frac{1}{\omega_{1234}} \right), \mathcal{P} \left(\frac{1}{-\omega_{2143}} \right), \mathcal{P} \left(\frac{1}{\omega_{3412}} \right), \mathcal{P} \left(\frac{1}{-\omega_{4321}} \right) \right] \otimes \mathbb{1}_{2 \times 2} \quad (165)$$

and

$$\delta(\underline{\omega}_{1234}) = \text{diag} \left[\delta(\omega_{1234}^{\text{abcd}}), \delta(-\omega_{2143}^{\text{abcd}}), \delta(\omega_{3412}^{\text{abcd}}), \delta(-\omega_{4321}^{\text{abcd}}) \right] \otimes \mathbb{1}_{2 \times 2}. \quad (166)$$

We obtain

$$\frac{\partial}{\partial t} W(p_1, t) = C_{\text{diss}}[W](p_1, t) + C_{\text{cons}}[W](p_1, t) \quad (167)$$

with

$$C_{\text{diss}}[W](p_1, t) = \frac{\pi}{(2\pi)^3} \int_{\mathbb{R}^{3d}} dp_{234} \delta(\underline{p}) \delta(\underline{\omega}_{1234}) (2 D[W]_{1234} + \mathcal{B}[W]_{1234} + \mathcal{B}[W]_{1234}^*) \quad (168)$$

and

$$C_{\text{cons}}[W](p_1, t) = \frac{i}{(2\pi)^3} \int_{\mathbb{R}^{3d}} dp_{234} \delta(\underline{p}) \mathcal{P}(\underline{\omega}_{1234}^{-1}) (\mathcal{B}[W]_{1234}^* - \mathcal{B}[W]_{1234}). \quad (169)$$

Note that without spin interaction the conservative part would vanish since \mathcal{B} and \mathcal{B}^* cancel out. Finally, algebraic reformulation and using the symmetry properties leads to

$$\mathcal{A}_{\text{quad}}[W]_{1234} + \mathcal{A}_{\text{tr}}[W]_{1234} = 2 D[W]_{1234} + \mathcal{B}[W]_{1234} + \mathcal{B}[W]_{1234}^* \quad (170)$$

and

$$[h_{\text{eff}}[W]_{1234}, W_1] = \mathcal{B}[W]_{1234}^* - \mathcal{B}[W]_{1234}. \quad (171)$$

Appendix B. Initial Wigner state $W(p, 0)$

For reproducibility, we record the analytical formula of the initial Wigner state $W(p, 0)$ used in the simulations (figure 1). We specify the state in dependence of the energy ε , which is related to the momentum via the dispersion relation $\varepsilon = \omega^\alpha(p)$ for particle type α , see equation (11).

The **a**-component is

$$\begin{aligned} W_{\uparrow\uparrow}^{\text{a}}(\varepsilon, 0) &= \frac{5}{2} e^{-2\varepsilon} \left(\varepsilon^2 + \frac{1}{4} \right)^2, \\ W_{\uparrow\downarrow}^{\text{a}}(\varepsilon, 0) &= 42 e^{2i(\varepsilon-1/3) - \frac{1}{2}(\varepsilon-15/4)^2 - 2\varepsilon}, \\ W_{\downarrow\downarrow}^{\text{a}}(\varepsilon, 0) &= \frac{1}{6} \text{erfc}(\varepsilon - 6) e^{-2\varepsilon/3} \text{atan}(\varepsilon + 1) \left(2 \text{erf}\left(\frac{\varepsilon}{2}\right) + \frac{1}{8} \right) \left(2 + \frac{1}{2} \sin(3\varepsilon) \right), \end{aligned} \quad (172)$$

the **b**-component reads

$$\begin{aligned} W_{\uparrow\uparrow}^{\text{b}}(\varepsilon, 0) &= \frac{2}{3} (2 + \sin(2\varepsilon)) (2 + \Gamma(1 + \varepsilon))^{-1}, \\ W_{\uparrow\downarrow}^{\text{b}}(\varepsilon, 0) &= \frac{1}{2} \zeta \left(\left(1 + \frac{i}{2} \right) \varepsilon \right) e^{-2\varepsilon}, \\ W_{\downarrow\downarrow}^{\text{b}}(\varepsilon, 0) &= e^{-(1+2\varepsilon/3)}, \end{aligned} \quad (173)$$

the c-component

$$\begin{aligned} W_{\uparrow\uparrow}^c(\varepsilon, 0) &= \frac{2}{3} \operatorname{erfc}\left(\frac{\varepsilon}{2}\right)\left(\varepsilon^2 + \frac{4}{5}\right)\left(\frac{3}{5} + \frac{1}{6}\varepsilon^2\right), \\ W_{\uparrow\downarrow}^c(\varepsilon, 0) &= \frac{1}{2} e^{-3\varepsilon/2}(1 + \operatorname{erf}(\varepsilon - 2)) \operatorname{erfc}(\varepsilon - 6) \left(\frac{2}{5} - i\varepsilon + 4(1 + i)\varepsilon \sin(\varepsilon)^2\right), \\ W_{\downarrow\downarrow}^c(\varepsilon, 0) &= \operatorname{erfc}(\varepsilon - 6) e^{-\varepsilon/2} \left(1 + \sin(\varepsilon)^2\right) \left(3 + \frac{3}{5}\varepsilon\right)^{-1}, \end{aligned} \quad (174)$$

and the d-component

$$\begin{aligned} W_{\uparrow\uparrow}^d(\varepsilon, 0) &= \frac{3}{4\pi} \operatorname{erfc}(\varepsilon - 7) e^{-\varepsilon/2} \operatorname{Si}\left(6\varepsilon + \frac{1}{2}\right), \\ W_{\uparrow\downarrow}^d(\varepsilon, 0) &= \frac{1}{24} \operatorname{erfc}(\varepsilon - 6) e^{i\pi 6/7 - 3\varepsilon/2} \sqrt{\varepsilon} \left(15 - 18\varepsilon + 3\varepsilon^2\right), \\ W_{\downarrow\downarrow}^d(\varepsilon, 0) &= \operatorname{Ai}(\varepsilon - 1). \end{aligned} \quad (175)$$

Here $\zeta(s)$ is the Riemann zeta function, $\operatorname{erf}(z)$ the error function, $\operatorname{erfc}(z)$ the complementary error function, $\operatorname{Si}(z)$ the sine integral function and $\operatorname{Ai}(x)$ the Airy function.

The off-diagonal entries $W_{\uparrow\downarrow}^\alpha(\varepsilon, 0)$ are respective complex conjugates of $W_{\downarrow\uparrow}^\alpha(\varepsilon, 0)$ since $W(\varepsilon, 0)$ is Hermitian.

References

- [1] Lee H, Cheng Y-C and Fleming G R 2007 Coherence dynamics in photosynthesis: protein protection of excitonic coherence *Science* **316** 1462–5
- [2] Collini E, Wong C Y, Wilk K E, Curmi P M G, Brumer P and Scholes G D 2010 Coherently wired light-harvesting in photosynthetic marine algae at ambient temperature *Nature* **463** 644–7
- [3] Maeda K, Henbest K B, Cintolesi F, Kuprov I, Rodgers C T, Liddell P A, Gust D, Timmel C R and Hore P J 2008 Chemical compass model of avian magnetoreception *Nature* **453** 387–90
- [4] Rodgers C T and Hore P J 2009 Chemical magnetoreception in birds: the radical pair mechanism *Proc. Natl Acad. Sci. USA* **106** 353–60
- [5] Tsymal E Y, Mryasov O N and LeClair P R 2003 Spin-dependent tunnelling in magnetic tunnel junctions *J. Phys.: Condens. Matter* **15** 109–42
- [6] Yoo J-W, Chen C-Y, Jang H W, Bark C W, Prigodin V N, Eom C B and Epstein A J 2010 Spin injection/detection using an organic-based magnetic semiconductor *Nat. Mater.* **9** 638–42
- [7] Liu Y, Cuellar F A, Sefrioui Z, Freeland J W, Fitzsimmons M R, Leon C, Santamaria J and te Velthuis S G E 2013 Emergent spin filter at the interface between ferromagnetic and insulating layered oxides *Phys. Rev. Lett.* **111** 247203
- [8] Archambault S *et al* 2009 Dark matter spin-dependent limits for WIMP interactions on ^{19}F by PICASSO *Phys. Lett. B* **682** 185–92
- [9] Fürst M L R, Mendl C B and Spohn H 2012 Matrix-valued Boltzmann equation for the Hubbard chain *Phys. Rev. E* **86** 031122
- [10] Fürst M L R, Mendl C B and Spohn H 2013 Matrix-valued Boltzmann equation for the non-integrable Hubbard chain *Phys. Rev. E* **88** 012108
- [11] Fermi E 1934 Versuch einer Theorie der β -Strahlen I *Z. Phys.* **88** 161–77
- [12] Spohn H 2006 Collisional invariants for the phonon Boltzmann equation *J. Stat. Phys.* **124** 1131–5
- [13] Semikoz D V and Tkachev I I 1997 Condensation of bosons in the kinetic regime *Phys. Rev. D* **55** 489–502
- [14] Zakharov V E 1967 Weak-turbulence spectrum in a plasma without a magnetic field *Sov. Phys.—JETP* **24** 455
- [15] Markowich P A and Pareschi L 2005 Fast conservative and entropic numerical methods for the Boson Boltzmann equation *Numer. Math.* **99** 509–32
- [16] Greiner W and Müller B 2010 *Gauge Theory of Weak Interactions* (Berlin: Springer)
- [17] Fürst M L R, Lukkarinen J, Mei P and Spohn H 2013 Derivation of a matrix-valued Boltzmann equation for the Hubbard model *J. Phys. A: Math. Theor.* **46** 485002