Generalized Functions

Christian Mendl

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We justify some special fundamental solutions to well known partial differential equations in the sense of generalized functions.

1 The potential equation

First we want to show the following key identity of electrostatics

$$-\triangle \frac{1}{4\pi r} = \delta(x), \quad r = |x|$$

in the sense of generalized functions.

Proof. Let

$$U(\psi) := \int \frac{\psi(x)}{4\pi r} \, \mathrm{d}x \qquad \forall\, \psi \in C_0^\infty(\mathbb{R}^3).$$

We have to proof that $-\triangle U = \delta$, i.e., by definition,

$$-U(\triangle\psi) = \psi(0) \qquad \forall\, \psi \in C_0^\infty(\mathbb{R}^3).$$

Our simple trick is replacing

$$\int_{\mathbb{R}^3} \mathrm{d}x \qquad \text{by} \qquad \lim_{\rho\to 0} \int_{G_\rho} \mathrm{d}x, \quad G_\rho := \mathbb{R}^3 \backslash K_\rho(0).$$

This is valid as integration is continuous with respect to the limits. Applying Green's identity

$$\int_{G_\rho} u\triangle v - v\triangle u \, \mathrm{d}x = \int_{\partial G_\rho} u\partial_\nu v - v\partial_\nu u \, \mathrm{d}S, \qquad (u, v \in C^2(G_\rho)) \tag{1}$$

we get

$$\int_{G_\rho} \frac{\triangle\psi}{4\pi r} \, \mathrm{d}x = \int_{G_\rho} \psi \underbrace{\triangle \frac{1}{4\pi r}}_{=0} \mathrm{d}x + \int_{\partial G_\rho} \frac{1}{4\pi r} \partial_\nu \psi - \psi \partial_\nu \frac{1}{4\pi r} \, \mathrm{d}S.$$

Observe that the outer unit normal points to the origin, so $\partial_\nu = -\partial_r$. Thus, the last integral becomes

$$\int_{r=\rho} \left(-\frac{1}{4\pi r} \frac{\partial\psi}{\partial r} - \frac{\psi}{4\pi r^2} \right) r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \to -\psi(0) \quad \text{as } \rho \to 0.$$

□

2 The heat equation

The heat equation generalizes nicely to n dimensions. Define the integral kernel

$$u(t,x) := \begin{cases} \frac{\mathrm{e}^{-r^2/4t}}{\sqrt{4\pi t}^n} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}, \qquad r = |x|$$

and let

$$U(\psi) := \int_{\mathbb{R}^n} \int_{\mathbb{R}} u(t,x)\psi(t,x)\,\mathrm{d}t\,\mathrm{d}x, \quad \psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n).$$

Then we have

$$\partial_t U - \triangle U = \delta(t,x).$$

Remark: as one might expect,

$$\partial_t u(t,x) = \left(\frac{r^2}{4t^2} - \frac{n}{2t}\right) u(t,x) = \triangle u(t,x). \tag{2}$$

Therefore, u is called the *fundamental solution* of the heat equation.

Proof. By definition, for all $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$,

$$\left(\partial_t U - \triangle U\right)(\psi) = \int_{R^n} \int_0^\infty u\,\left(-\partial_t \psi - \triangle \psi\right)\,\mathrm{d}t\,\mathrm{d}x.$$

Integration by parts yields

$$\int_0^\infty u\,\left(-\partial_t \psi\right)\,\mathrm{d}t = -u\,\psi\Big|_0^\infty + \int_0^\infty \partial_t u\,\psi\,\mathrm{d}t.$$

The boundary term vanishes as ψ is zero outside a bounded set and $\lim_{t\to 0} u(t,x) = 0$ (except $x = 0$; remember that $\{0\} \subset \mathbb{R}^n$ has measure zero). The statement to be proven now reads

$$\int_{R^n} \int_0^\infty \partial_t u\,\psi - u\,\triangle\psi\,\mathrm{d}t\,\mathrm{d}x = \psi(0) \qquad \forall\,\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n).$$

Introduce $G_\rho := \mathbb{R}^n \backslash K_\rho(0)$ as above. Since $u\,\triangle\psi \in C_0(\mathbb{R} \times G_\rho)$, Fubini's theorem tells us that

$$\int_{G_\rho} \int_0^\infty -u\,\triangle\psi\,\mathrm{d}t\,\mathrm{d}x = \int_0^\infty \int_{G_\rho} -u\,\triangle\psi\,\mathrm{d}x\,\mathrm{d}t.$$

By Gauss' integral theorem,

$$\int_{G_\rho} -u\,\triangle\psi\,\mathrm{d}x = \int_{\partial G_\rho} -u\,\nabla\psi \cdot \nu\,\mathrm{d}x + \int_{\partial G_\rho} \psi\,\nabla u \cdot \nu\,\mathrm{d}x - \int_{G_\rho} \triangle u\,\psi\,\mathrm{d}x.$$

Applying Fubini's theorem again, together with (2) we have

$$\int_{\mathbb{R}^n} \int_0^\infty \partial_t u\,\psi\,\mathrm{d}x - \lim_{\rho\to 0} \int_0^\infty \int_{G_\rho} \triangle u\,\psi\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^n} \int_0^\infty \left(\partial_t u - \triangle u\right)\,\psi\,\mathrm{d}t\,\mathrm{d}x = 0,$$

so what remains to be shown is

(i) $\lim_{\rho\to 0}\int_0^\infty \int_{\partial G_\rho} -u\,\nabla\psi\cdot\nu\,\mathrm{d}S\,\mathrm{d}t = 0$

(ii) $\lim_{\rho\to 0}\int_0^\infty \int_{\partial G_\rho} \psi\,\nabla u\cdot\nu\,\mathrm{d}S\,\mathrm{d}t = \psi(0)$.

Ad (i). u only depends on t and $|x|$, so we can write $u(t,r)$ instead of $u(t,x)$. Remember that the unit normal ν of ∂G_ρ points towards the origin; this yields

$$\int_{\partial G_\rho} -u\,\nabla\psi\cdot\nu\,\mathrm{d}S = u(t,\rho)\int_{\partial G_\rho} \nabla\psi\cdot\frac{x}{r}\,\mathrm{d}S = u(t,\rho)\rho^{n-1}\chi(t,\rho),$$

$$\chi(t,\rho) := \int_{|y|=1} \nabla\psi(t,\rho y)\cdot y\,\mathrm{d}S \;\in C_0^\infty\left(\mathbb{R}\times(0,\infty)\right).$$

Using the Taylor expansion

$$\nabla\psi(t,\rho y) = \nabla\psi(t,0) + \mathcal{O}(\rho) \quad \text{and}$$
$$\int_{|y|=1} \nabla\psi(t,0)\cdot y\,\mathrm{d}S = 0,$$

we have

$$\lim_{\rho\to 0}\chi(t,\rho) = 0 \quad \text{equicontinuously in } t. \tag{3}$$

Choose such a fixed $T>0$ that $\chi(t,\rho)=0$ for all $t>T, \rho\in(0,\infty)$. Applying a change of variables, $s=\frac{\rho}{\sqrt{4t}}$,

$$\int_0^\infty u(t,\rho)\rho^{n-1}\chi(t,\rho)\,\mathrm{d}t = \frac{\rho}{2\sqrt{\pi}^n}\int_{\frac{\rho}{\sqrt{4T}}}^\infty s^{n-3}\mathrm{e}^{-s^2}\chi\left((\rho/2s)^2,\rho\right)\,\mathrm{d}s.$$

Evidently, since χ is bounded, this will converge towards 0 as $\rho\to 0$ if $n\geq 3$. The cases $n=1$ and $n=2$ are subtler. By the fact that $\psi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ and (3), there exists a continuous function $M(\rho)$ with

$$\left|\frac{\chi(t,\rho)}{2\sqrt{\pi}^n}\right| \leq M(\rho) \quad \forall t\in\mathbb{R},\ \rho>0.$$

Therefore the absolete value of the last integral is less or equal

$$M(\rho)\,\rho\int_{\frac{\rho}{\sqrt{4T}}}^\infty s^{n-3}\mathrm{e}^{-s^2}\,\mathrm{d}s.$$

Now let $n=1$. Then a majorant is given by

$$M(\rho)\,\rho\int_{\frac{\rho}{\sqrt{4T}}}^\infty \frac{1}{s^2}\,\mathrm{d}s = \sqrt{4T}M(\rho)\to 0 \text{ as } \rho\to 0.$$

Similarly for $n=2$:

$$M(\rho)\,\rho\left(\int_{\frac{\rho}{\sqrt{4T}}}^1 \frac{1}{s}\,\mathrm{d}s + \int_1^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s\right)\to 0 \text{ as } \rho\to 0.$$

Ad (ii). We have

$$\nabla u\cdot\nu = -\partial_r u = \frac{r}{2t}u, \text{ so}$$

$$\int_0^\infty \int_{\partial G_\rho} \psi\, \nabla u \cdot \nu \,\mathrm{d}S\,\mathrm{d}t = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{\rho}{t} \frac{1}{\sqrt{4t}^n} \mathrm{e}^{-\rho^2/4t} \rho^{n-1} \chi(t,\rho)\,\mathrm{d}t,$$

$$\chi(t,\rho) := \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}^n} \int_{|y|=1} \psi(t,\rho y)\,\mathrm{d}S \,\in C_0^\infty\left(\mathbb{R}\times(0,\infty)\right).$$

Since the surface area of the unit sphere in n dimensions is $\frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)}$, $\chi(t,0) = \psi(t,0)$. As above, we change variables by letting $s = \frac{\rho}{\sqrt{4t}}$ and get

$$\frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty 2s^{n-1}\mathrm{e}^{-s^2} \chi\left((\rho/2s)^2, \rho\right)\,\mathrm{d}s.$$

By the theorem of dominated convergence, as $\rho \to 0$, this converges towards

$$\frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty 2s^{n-1}\mathrm{e}^{-s^2} \chi(0,0)\,\mathrm{d}s = \chi(0,0) = \psi(0).$$

□

3 The Helmholtz equation

Let $k \in \mathbb{R}$. Then

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right) \frac{\mathrm{e}^{ik|x|}}{2ik} = \delta(x), \quad x \in \mathbb{R}.$$

Using integration by parts, this follows from the decomposition

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{ik|x|}}{2ik}\left(\psi'' + k^2\psi\right)\,\mathrm{d}x = \int_0^\infty \frac{\mathrm{e}^{ikx}}{2ik}\psi''\,\mathrm{d}x + \int_{-\infty}^0 \frac{\mathrm{e}^{-ikx}}{2ik}\psi''\,\mathrm{d}x + \int_{\mathbb{R}} \frac{k}{2i}\mathrm{e}^{ik|x|}\psi\,\mathrm{d}x,$$

where $\psi \in C_0^\infty(\mathbb{R})$. In 3 dimensions,

$$\left(\triangle + k^2\right) \frac{-\mathrm{e}^{\pm ikr}}{4\pi r} = \delta(x), \quad x \in \mathbb{R}^3.$$

Proof. We have to show that

$$\int \frac{\mathrm{e}^{\pm ikr}}{4\pi r}\left(\triangle + k^2\right)\psi\,\mathrm{d}x = -\psi(0) \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3).$$

We set $G_\rho := \mathbb{R}^3 \backslash K_\rho(0)$ as above. By the same arguments,

$$\int \frac{\mathrm{e}^{\pm ikr}}{4\pi r}\left(\triangle + k^2\right)\psi\,\mathrm{d}x = \lim_{\rho\to 0} \int_{G_\rho} \frac{\mathrm{e}^{\pm ikr}}{4\pi r}\left(\triangle + k^2\right)\psi\,\mathrm{d}x.$$

Using Green's identity (1),

$$\int_{G_\rho} \frac{\mathrm{e}^{\pm ikr}}{4\pi r} \triangle\psi\,\mathrm{d}x = \int_{G_\rho} \psi\, \triangle \frac{\mathrm{e}^{\pm ikr}}{4\pi r}\,\mathrm{d}x + \int_{\partial G_\rho} \frac{\mathrm{e}^{\pm ikr}}{4\pi r}\partial_\nu\psi - \psi\,\partial_\nu \frac{\mathrm{e}^{\pm ikr}}{4\pi r}\,\mathrm{d}S.$$

In the second integral we need the Laplacian in spherical coordinates:

$$\triangle \frac{\mathrm{e}^{\pm ikr}}{4\pi r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}\frac{\mathrm{e}^{\pm ikr}}{4\pi} = -k^2\,\frac{\mathrm{e}^{\pm ikr}}{4\pi r}.$$

As $\partial_\nu = -\partial_r$, the last integral is equal to

$$\int_{r=\rho} \left(-\frac{1}{4\pi r}\frac{\partial \psi}{\partial r} \pm \frac{ik}{4\pi r}\psi - \frac{1}{4\pi r^2}\psi \right) \mathrm{e}^{\pm ikr} r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \to -\psi(0) \quad \text{as } \rho \to 0.$$

The asserting now follows by putting everything together. □

We may use Fourier transformation to find the above Green functions. In the following section, we restrict ourselves to formal arguments. Let's start with

$$\left(\triangle + k^2\right) G(x) = \delta(x), \quad x \in \mathbb{R}^3.$$

Integration by parts yields

$$\begin{aligned}\left(-q^2 + k^2\right)(\mathrm{F}G)(q) &= (2\pi)^{-\frac{3}{2}} \int G(x)\left(-q^2+k^2\right) \mathrm{e}^{-iq\cdot x}\,\mathrm{d}x \\ &= (2\pi)^{-\frac{3}{2}} \int \mathrm{e}^{-iq\cdot x}\left(\triangle + k^2\right) G(x)\,\mathrm{d}x = (2\pi)^{-\frac{3}{2}}.\end{aligned}$$

Now we use the inverse Fourier transform:

$$G(x) = \frac{1}{(2\pi)^3}\int \frac{1}{k^2 - q^2}\mathrm{e}^{iq\cdot x}\,\mathrm{d}q.$$

In spherical coordinates, this equals

$$\frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{k^2-q^2}\,\mathrm{e}^{iqr\cos\theta} q^2 \sin\theta \,\mathrm{d}q\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{4\pi^2 ir}\int_{-\infty}^{\infty} \frac{q\mathrm{e}^{iqr}}{k^2-q^2}\,\mathrm{d}q.$$

Unfortunately, this integral doesn't exist, so we shift the poles by an infinitesimal amount ϵ. Now applying the residue theorem, we get

$$\frac{1}{4\pi^2 ir}\int_{-\infty}^{\infty} \frac{q\mathrm{e}^{iqr}}{k^2-q^2 \pm i\epsilon}\,\mathrm{d}q = -\frac{\mathrm{e}^{\pm ikr}}{4\pi r}.$$

4 The wave equation

Let

$$U(\psi) := \int_0^\infty \frac{1}{4\pi t}\int_{|x|=t} \psi(t,x)\,\mathrm{d}S\,\mathrm{d}t, \quad \psi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3).$$

Then

$$\partial_t^2 U - \triangle U = \delta(t,x)$$

in the sense of generalized functions.

Proof. We have

$$U(\psi) = \int_0^\infty \int_{|x|=t} \frac{1}{4\pi r}\psi(r,x)\,\mathrm{d}S\,\mathrm{d}t = \int_{\mathbb{R}^3} \frac{1}{4\pi r}\psi(r,x)\,\mathrm{d}x, \quad r = |x|.$$

Let $\phi(x) := \psi(r,x)$. Note that in general, $\phi \notin C_0^\infty(\mathbb{R}^3)$, but $\phi \in C_0^\infty(G_\rho)$. Literally following the proof of the potential equation, we get

$$-U(\triangle\phi) = \phi(0) = \psi(0).$$

A short calculation shows that

$$\triangle\phi(x) = \psi_{tt}(r,x) + 2\,\psi_{t\nabla}(r,x)\cdot e_r + \frac{2}{r}\,\psi_t(r,x) + \psi_\triangle(r,x).$$

Here, ψ_t denotes the partial derivative with respect to t, i.e.

$$\psi_t(t,x) \equiv \partial_t\psi(t,x), \quad \psi_\nabla(t,x) \equiv \nabla\psi(t,x), \quad \psi_\triangle(t,x) \equiv \triangle\psi(t,x).$$

Putting everything together, we have

$$\begin{aligned}&\left(\partial_t^2 U - \triangle U\right)(\psi) = U\left(\psi_{tt} - \triangle\psi\right)\\ &= \int_{\mathbb{R}^3} \frac{1}{4\pi r}\left[-\triangle\phi(x) + 2\left(\psi_{tt}(r,x) + \psi_{t\nabla}(r,x)\cdot e_r + \frac{1}{r}\psi_t(r,x)\right)\right]\mathrm{d}x\\ &= \psi(0) + 2\int_0^\infty \frac{t}{4\pi}\int_{|y|=1} \psi_{tt}(t,ty) + \psi_{t\nabla}(t,ty)\cdot y + \frac{1}{t}\psi_t(t,ty)\,\mathrm{d}S\,\mathrm{d}t,\end{aligned}$$

so what remains to be shown is that the last integral is zero. In fact, by letting

$$\chi(t) := \frac{1}{4\pi}\int_{|y|=1} \psi_t(t,ty)\,\mathrm{d}S \;\in C_0^\infty\left((0,\infty)\right),$$

it equals

$$\int_0^\infty t\cdot\chi'(t) + \chi(t)\,\mathrm{d}t = t\cdot\chi(t)\Big|_0^\infty - \int_0^\infty \chi(t)\,\mathrm{d}t + \int_0^\infty \chi(t)\,\mathrm{d}t = 0.$$

□

References

[1] Eberhard Zeidler: *Applied Functional Analysis (Applications to Mathematical Physics)*, AMS Vol. 108